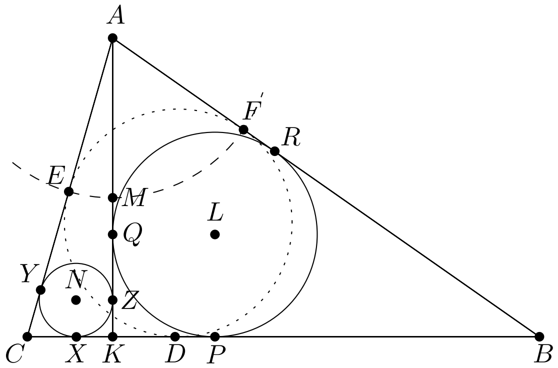
Math match Good Water 2010

Solutions

Problem 1. **(Incircles)** (Mathematical Reflections 2010, (5)) The incircle of triangle ABC touches sides BC, CA, AB at D, E, F, respectively. Let K be a point on side BC and let M be the point on the line segment AK such that $AM = AE = AF$. Denote by L and N the incenters of triangles ABK and ACK, respectively. Prove that K is the foot of the altitude from A if and only if $DLMN$ is a square.

Solution. Assume first that $DLMN$ is a square. Note that KL and KN bisect angles AKC and AKB so $\angle LKN = 90^\circ = \angle LDN$ and K lies on the circumcircle of $DLMN$. This implies $\angle MKD = \angle MLD = 90^\circ$ and AK is indeed an altitude in $\triangle ABC$.

Now assume K is the foot of the altitude. Denote by P, Q, R the points where the incircle of ABK touches KB, AK, BA, respectively and denote by X, Y, Z the points where ACK touches the sides KC, CA, AK.



Note that $KZNX$ and $KPLQ$ are squares. We use equal tangents to get

$$XD - PD = EY - FR = AY - AR = AZ - AQ = KQ - KZ = KP - KX,$$

which implies $XK = DP$ or $XN = DP$. Similarly we can show that $XD = LP$ and this gives us $\triangle NXD \cong \triangle DPL$ (SAS) and so

$$\angle LDN = 180^\circ - \angle LDP - \angle NDX = 180^\circ - \angle LDP - \angle DLP = 90^\circ$$

and $\triangle DNL$ is right and isosceles.

Now as N lies on the bisector of CAK and $AM = AE$, the quadrilateral $AENM$ is a kite and we have $NM = NE$. Similarly N lies on bisector of $\triangle ACB$ and $CE = CD$ so we get $ND = NE = NM$. Analogous argument shows $LM = LD$ implying the desired result.

Problem 2. **(Polynomials)**(Czech and Slovak 2001) Find all polynomials P such that

$$P(x)^2 + P(-x) = P(x^2) + P(x)$$

holds for every $x \in \mathbb{R}$.

Solution. Adding $P(-x)$ to both sides of the equation we make the right handside an even function. Hence the left handside is also an even function. Rewriting this we get

$$P(x)^2 + 2P(-x) = P(-x)^2 + 2P(x) \quad \Leftrightarrow \quad (P(x) - P(-x))\,(P(x) + P(-x) - 2) = 0.$$

So in one case we obtain $P(x) = P(-x)$ which compared with the original equation gives $P(x)^2 = P(x^2)$. The second case is $P(x) + P(-x) = 2$ or $Q(x) = -Q(-x)$ where $Q(x) = P(x) - 1$. Plugging this in the original equation yields $Q(x)^2 = Q(x^2)$.

Either way we need to find all polynomials satisfying $R(x^2) = R(x)^2$. Obviously $R(x) \equiv 0, 1$ are the only constant solutions so from now on we may assume R is nonconstant. Let

$$R(x) = ax^n + bx^k + S(x)$$

for some $k, n \in \mathbb{N}_0, n > k$, $a \neq 0$ and some polynomial $S(x)$ of degree less than k. Assume $b \neq 0$. Now equating coefficients of x^{n+k} we obtain $2ab = 0$, which is a contradiction so the only polynomials with the desired property are x^k with $k \in \mathbb{N}_0$.

Putting all this together we may conclude that $P(x)$ must be of one of the following forms: $x^2, x^4, \ldots,$ $x+1, x^3+1, x^5+1\ldots$ and the constant polynomials $0, 1$. All these polynomials are indeed solutions so we are done.

Problem 3. **(Set)**(China MO training 1988) Determine the smallest value of the natural number $n > 3$ with the property that whenever the set $S_n = 3, 4, ..., n$ is partitioned into the union of two subsets, at least one of the subsets contains three numbers a, b and c (not necessarily distinct) such that $ab = c$.

Solution. We first show that $3^5 = 243$ has the property, then we will show it is the least solution.

Suppose S_{243} is partitioned into two subsets X_1, X_2. Without loss of generality, let 3 be in X_1. If $3^2 = 9$ is in X_1, then we are done. Otherwise, 9 is in X_2. If $9^2 = 81$ is in X_2, then we are done. Otherwise, 81 is in X_1. If $81/3 = 27$ is in X_1, then we are done. Otherwise, 27 is in X_2. Finally, either $3 \cdot 81 = 243$ is in X_1 or $9 \cdot 27 = 243$ is in X_2. In either case we are done.

To show 243 is the smallest, we will show that S_{242} can be partitioned into two subsets, each of which does not contain products of its elements. It is easy to see that $X_1 = \{9, 10, \ldots 80\}$ and $X_2 = S_{242} \setminus X_1$ work so we are done.

Problem 4. **(Pedal triangle)**(Mathematical Reflections 2010 (5)) Let P be a point inside triangle ABC and let d_a, d_b, d_c be the distances from point P to the sides of the triangle. Prove that

$$\frac{K}{d_a d_b d_c} \geq \frac{s}{2Rr}$$

where K is the area of the pedal triangle of P and s, R, r are the semiperimeter, circumradius, and inradius of triangle ABC.

Solution. Denote by X, Y, Z the projections of P onto the sides BC, CA, AB, respectively. First we note that $K = K[PXY] + K[PYZ] + K[PZX]$ so we can write

$$2K = d_a d_b \sin\gamma + d_b d_c \sin\alpha + d_c d_a \sin\beta.$$

Plugging in and using the law of sines we get

$$2 \cdot LHS = \frac{\sin\alpha}{d_a} + \frac{\sin\beta}{d_b} + \frac{\sin\gamma}{d_c} = \frac{1}{2R}\left(\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c}\right).$$

Next we observe that $K[ABC] = K[ABP] + K[BCP] + K[CAP]$ and this yields

$$ad_a + bd_b + cd_c = 2K[ABC] = 2rs.$$

Finally we use the Cauchy inequality to get

$$LHS = \frac{1}{4R}\left(\frac{a}{d_a} + \frac{b}{d_b} + \frac{c}{d_c}\right) \geq \frac{s^2}{R(ad_a + bd_b + cd_c)} = \frac{s}{2Rr},$$

which concludes the proof.

Problem 5. **(Table)**(Russia 1995) Is it possible to fill in the cells of a 9×9 table with positive integers ranging from 1 to 81 in such a way that the sum of the elements of every 3×3 square is the same?

Solution. Place $0, 1, 2, 3, 4, 5, 6, 7, 8$ on the first, fourth and seventh rows. Place $3, 4, 5, 6, 7, 8, 0, 1, 2$ on the second, fifth and eigth rows. Place $6, 7, 8, 0, 1, 2, 3, 4, 5$ on the third, sixth and ninth rows. Then every 3×3 square has sum 36. Consider this table and its $90°$ rotation. For each cell, fill it with the number $9a + b + 1$, where a is the number in the cell originally and b is the number in the cell after the table is rotated by $90°$. By inspection, 1 to 81 appears exactly once each and every 3×3 square has sum $9 \times 36 + 36 + 9 = 369$.

Problem 6. (Square)(1969 Kürschak Eötvos Hungary) Let n be a positive integer. Show that if

$$2 + 2\sqrt{28n^2 + 1}$$

is an integer, then it is a square.

Solution. If $2 + 2\sqrt{28n^2 + 1} = m$, an integer, then $4(28n^2 + 1) = (m - 2)^2$. This implies m is even, say $m = 2k$. So $28n^2 = k^2 - 2k$. This implies k is even, say $k = 2j$. Then $7n^2 = j(j - 1)$. Since $\gcd(j, j - 1) = 1$, either $j = 7x^2, j - 1 = y^2$ or $j = x^2, j - 1 = 7y^2$. In the former case, we get $-1 \equiv y^2 \pmod 7$, which is impossible. In the latter case, $m = 2k = 4j = 4x^2$ is a square.