The Graph Theory

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Part 1: Definitions

You can imagine a *graph* G as some points in a plane (we call them the *vertices*), some of which are connected by lines (the *edges*).

Definition. *A simple graph is an ordered pair (V, E), where V is a finite set of vertices and E is a set of unordered pairs of elements of V, i.e. $E \subseteq \binom{V}{2}$. We often denote the edge $\{u, v\}$ simply by uv.*

We say that a simple graph has no loops (i.e. edges both ends of which are in the same vertex) and no multiple edges (i.e. no two points are connected by more than one edge). A graph which contains multiple edges is called a multigraph, a graph with multiple edges and loops is called a pseudograph.

A directed graph (a digraph) is an ordered pair (V, E), where V is a finite set of vertices and E is a set of ordered pairs of the elements of V, i.e. $E \subseteq V^2$. We call the edges of a digraph the arcs; by uv we will mean the edge $[u, v]$ – be careful to distinguish it from vu!

An oriented graph is a digraph that we get by orienting the edges of a simple graph. The underlying graph of a digraph is obtained by replacing the arcs by edges.

Part 2: Degrees

Two vertices a, b are *adjacent* to each other if they are joined by an edge; then they are *incident* with the edge ab which joins them. The *degree* of a vertex v is the number of edges meeting at v; it is denoted by *deg* v. The *degree sequence* of a graph is a list of the degrees of the vertices from the smallest to the biggest.
In a digraph, the *outdegree* of a vertex v is the number of edges that leave it, i.e. the number of edges of the form $vu, u \in V$, and it's *indegree* is the number of edges that enter it, i.e. the number of edges of the form $uv, u \in V$. The edge vu is *incident from* the vertex v and *incident to* the vertex u. We can also define the *outdegree sequence* and the *indegree sequence*.

Lemma. (The handshaking) *In any graph G, the sum of the degrees of the vertices is equal to twice the number of edges:*

$$\sum_{v \in V} deg\ v = 2 \cdot e.$$

Corollary. *In any graph, the number of vertices of odd degree is even.*

Lemma. (The handshaking, for digraphs)

$$\sum_{v \in V} outdeg\ v = \sum_{v \in V} indeg\ v.$$

Theorem. (The degree sequence theorem) *Let $D = (d_1, d_2, \ldots, d_n)$ be a sequence of natural numbers such that $d_1 \le d_2 \le \cdots \le d_n$. Let $E = (e_1, e_2, \ldots, e_n)$ be defined by $e_i = d_i, i < n - d_n$, $e_i = d_i - 1, i > n - d_n$. Then D is a degree sequence iff E is a degree sequence.*

To determine, whether a sequence is a degree sequence, repeat the reduction according to the degree sequence theorem repeatedly.

Part 3: Connectivity

Definition. *A walk of length k in a graph G is a succession of k edges of G in the form uv, vw, ... , yx. Iff in a walk all the edges are different, we call it a trail. Iff all the vertices are different, we call it a path. A graph G is called connected iff there is a path between any pair of its vertices. Otherwise it is called disconnected; a disconnected graph can be split up into a number of connected subgraphs called the components.*

Lemma. *Any walk joining the vertices u and v can be shortened into a path which joins the same vertices.*

If we define an equivalence relation $\sim$ on the vertices of a graph G so that $u \sim v$ iff there is a path in G joining u and v, then the equivalence classes of $\sim$ are the components of G.

Part 4: Isomorphisms of graphs

Two graphs G and H are *isomorphic* iff H can be obtained from G by relabeling the vertices so that there is a one-to-one correspondence between the vertices of G and those of H such that the number of edges joining any pair of vertices in G is equal to the number of edges joining the corresponding pair of vertices in H.

Dictionary

graph – graf
  simple – prostý
  oriented, digraph – orientovaný
  connected – souvislý
  component – komponenta
edge – hrana
  arc – orientovaná, šipka
  loop – smyčka
  multiple – násobný
vertex (pl. vertices) – vrchol, uzel
  degree – stupeň
    sequence – skóre
    outdegree – výstupní stupeň
    indegree – vstupní stupeň
theorem – věta
walk – sled
trail – tah
path – cesta
iff = if and only if – právě tehdy, když
isomorphism – izomorfismus