Arrangements

4TH AUTUMN SERIES — DATE DUE: 6TH JANUARY 2025

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)

There are n pigs standing in a line. Among those, however, Matouš, Matěj and Michal do not want to stand next to each other[1]. Find the number of possible ways to arrange the pigs that satisfy this condition.

PROBLEM 2. (3 POINTS)

Sylva found a clock that had its numbers rearranged. For each of the twelve neighbouring pairs of numbers, she wrote down their sum. She then replaced each of these twelve sums with its remainder when divided by 13. Finally, she summed the twelve remainders. What is the smallest value Sylva could have obtained out of all possible arrangements?

PROBLEM 3. (3 POINTS)

Štepi is playing with a grid of 2024×2024 points. For every ordered triplet of distinct points (A, B, C) of the grid, he measures[2] and writes down $\angle ABC$. What is the average of all the numbers Štepi writes down?

PROBLEM 4. (5 POINTS)

Let n be a positive integer such that its base-10 representation contains each of the digits 0, 1, 2 and 3 at least once. Show that the digits of n can be permuted so that the new number[3] is divisible by 7.

PROBLEM 5. (5 POINTS)

Vítek owns a deck of 2024 cards, each of which has one of four suites. Initially, the deck is sorted in such a way that any four consecutive cards are of four different suites. Vítek then takes some consecutive block of cards from the top of the deck, reverses its order and inserts it back somewhere into the deck. Afterwards, Vítek separates the deck into quadruplets, consisting of the first through fourth card, then fifth through eighth, etc. Show that each of these quadruplets contains cards of four distinct suites.

PROBLEM 6. (5 POINTS)

Majda and Vašek are playing a game, in which Majda takes the first turn and then they alternate. Initially, the numbers 2000, 1999, ..., 3, 2, 1 are written on a board in this order. During his turn, Vašek can choose 1000 numbers and rearrange them as he wishes. Majda can, during her turn, choose k numbers and rearrange them, where k is a fixed positive integer. Majda wins if the numbers on the board are in the order 1, 2, 3, ..., 1999, 2000. What is the smallest k for which Majda can always win (regardless of how Vašek plays) after a finite number of turns?

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[1] Matouš, Matěj and Michal are pigs.

[2] Of the two angles determined by rays BA and BC, Štepi always measures the smaller one.

[3] We allow the new number to begin with zeroes.

PROBLEM 7. (5 POINTS)

Let p be an odd prime number and S_p be the set of permutations of the set $\{1, 2, \ldots, p\}$. For any $\pi \in S_p$, define $\Phi(\pi)$ as the number of multiples of p among the numbers

$$\pi(1), \qquad \pi(1) + \pi(2), \qquad \ldots, \qquad \pi(1) + \pi(2) + \cdots + \pi(p).$$

Find the value of $\displaystyle\frac{1}{p!} \sum_{\pi \in S_p} \Phi(\pi)$.

PROBLEM 8. (5 POINTS)

Matěj and Daník are standing in (not neccessarily the same) vertices of the complete graph on n vertices.[4] Each edge of this graph has a price, which is a nonnegative real number that has to be paid when moving along this edge, and each price is unique. Both Daník and Matěj make a journey that visits each vertex exactly once, according to the following rules:

(1) Daník likes expensive things, so at each step, he moves along the edge which costs the most among the ones leading to vertices he hasn't visited yet.
(2) Matěj likes cheap things, so at each step, he moves along the edge which costs the least among the ones leading to vertices he hasn't visited yet.

Show that in the end, Daník's total expenses are greater than or equal to Matěj's expenses.

[4] To learn what a *graph* or a *complete graph* is, see this older introductory text: *https://prase.cz/archive/42/uvod4p.pdf.*