Circles

4. PODZIMNÍ SÉRIE TERMÍN ODESLÁNÍ: 6. LEDNA 2020

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

ÚLOHA 1. (3 BODY)

Kua would like to draw 7 circles on a piece of paper so that every two circles intersect at two distinct points. And he would love it if there were exactly 7 points, in which 2 or more circles intersect. Show him how this is possible.

ÚLOHA 2. (3 BODY)

Let ω_1, ω_2 be two circles with centres O_1, O_2 intersecting at points X, Y such that $\angle O_1XO_2 = 90^\circ$. Let D be the intersection of O_1O_2 and ω_1 such that O_1 lies between D and O_2. Let P be the intersection of DX and ω_2 distinct from X. Prove that PO_2 is perpendicular to O_1O_2.

ÚLOHA 3. (3 BODY)

Jáchym drew three circles on a whiteboard. The circles had radii 2, 3, and 3 and each two were externally tangent. Then he drew the circle ω that is internally tangent to all three of them. Help him calculate the radius of ω.

ÚLOHA 4. (5 BODŮ)

Let $ABCD$ be a cyclic quadrilateral with circumcentre O such that AC and BD are perpendicular. Let ω_1, ω_2, ω_3, and ω_4 be circles, where the diameters of these circles are AO, BO, CO, and DO respectively. Finally, let P, Q, R, and S be the intersections of ω_1 with ω_2, ω_2 with ω_3, ω_3 with ω_4, and ω_4 with ω_1 respectively, distinct from O. Prove that $PQRS$ is a rectangle.

ÚLOHA 5. (5 BODŮ)

Let ω_1 and ω_2 be two circles externally tangent at T. Let C be a point on ω_2 such that the tangent at C intersects ω_1 at two distinct points X and Y. Now define P as the intersection of CT and ω_1 distinct from T. Show that PXY is an isosceles triangle.

ÚLOHA 6. (5 BODŮ)

There are $2n$ points on a circle labeled $1, 2, \ldots, 2n$ in some order. We define a *pairing* as a set of n segments between these points such that every point is an endpoint of exactly one of the segments. For a segment connecting points labelled a and b, we say its *value* is the number $|a - b|$. Finally, we say a pairing is *good*, if the sum of values of all n segments is equal to n^2. Show that for any initial order of labels there exists a good pairing such that no two segments intersect.

ÚLOHA 7. (5 BODŮ)

Let $ABCD$ be a parallelogram such that $\angle DAB$ is obtuse. Then, let M be the midpoint of AB and E be the intersection of the circumcircle of DAB and the line DM distinct from D. Finally, let H be the point on DA such that $\angle AHB = 90^\circ$. Prove that C, D, H, and E are concyclic.

ÚLOHA 8. (5 BODŮ)

Let ABC be a non-equilateral triangle and G its centroid. Denote the midpoints of line segments AB, CA, BC, AG, CG, and BG by M_C, M_B, M_A, N_A, N_C, and N_B respectively. Show that the circumcircles of $M_CN_AM_B$, $M_BN_CM_A$, and $M_AN_BM_C$ all intersect in a single point.