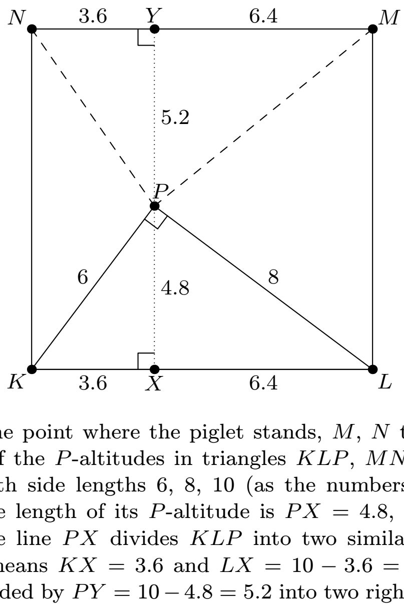
Distances

4^{TH} AUTUMN SERIES — MODEL SOLUTIONS

Problem 1.

There is a piglet standing inside a square field of size 10×10. *In each corner of the field, there is a goat observing the piglet. The piglet's distance to some two of the goats is* 6 *and* 8, *respectively. Find its distance to the remaining two goats.* *(Marian Poljak)*

SOLUTION:
Let us denote the goats, whose distances from the piglet are 6 and 8, by K and L respectively. There are two possibilities, the goats either stand in adjacent or opposite corners. But since the length of the diagonal is $\sqrt{2} \cdot 10$, which is more than $14 = 6 + 8$, we can rule out the latter option by the triangle inequality.



Furthermore, let P be the point where the piglet stands, M, N the remaining vertices of the square and X, Y the feet of the P-altitudes in triangles KLP, MNP respectively. The triangle KLP is a right triangle with side lengths 6, 8, 10 (as the numbers satisfy the converse of the Pythagorean Theorem). The length of its P-altitude is $PX = 4.8$, because the area of KLP is $\frac{1}{2}(6 \cdot 8) = \frac{1}{2}(PX \cdot 10)$. The line PX divides KLP into two similar triangles KXP and PXL (congruent angles), which means $KX = 3.6$ and $LX = 10 - 3.6 = 6.4$. Let us now look at the triangle MNP, which is divided by $PY = 10 - 4.8 = 5.2$ into two right triangles. We are interested in their hypotenuses. Using the Pythagorean theorem we get

$$MP = \sqrt{5.2^2 + 6.4^2} = \sqrt{68} = 2\sqrt{17},$$
$$NP = \sqrt{5.2^2 + 3.6^2} = \sqrt{40} = 2\sqrt{10}.$$

The distances from the remaining two goats are $2\sqrt{17}$ and $2\sqrt{10}$.

Poznámky:
Většina si s úlohou hravě poradila, někteří ale zapomněli vyloučit možnost, že kozy stojí v protilehlých vrcholech čtverce. Několik řešitelů využívalo Euklidovy věty o odvěsnách, dejte ale pozor, že v anglické literatuře se pod názvem *Euclid's theorem* myslí důkaz toho, že je prvočísel nekonečně mnoho. Také není dobré bezhlavě zaokrouhlovat, výsledek ve tvaru jako výše je naprosto v pořádku, ale 6.3 a 8.2 není, nicméně body jsem za to tentokrát nestrhávala, pokud byl postup v pořádku. *(Hedvika Ranošová)*

Problem 2.

There is an odd number (at least three) of politicians standing on a meadow so that no two pairs of politicians are the same distance apart. Each politician has an egg. Suddenly, each of them throws his egg and hits the face of the politician standing closest to him. Show that afterwards, there is a politician that does not have egg on his face. *(Josef Minařík)*

Solution:
The number of politicians and eggs is the same. Therefore, if there is a politician who was hit by more than one egg, there must also be a politician without egg on his face. Consider the pair of politicians with minimal distance between them. They will throw eggs at each other. If someone else throws an egg at one of them, we are done. So we can assume no one else throws an egg at them, and we can ignore this pair of politicians. The number of politicians is still odd since it decreased by 2. We can repeat this until only three politicians remain. In this case, the closest two will throw eggs at each other, so the last one remains without egg on his face.

Poznámky:
Většina řešitelů postupovala podobně jako vzorové řešení. Jiná řešení si všimla, že politici házející vejce tvoří permutaci, kterou lze rozložit na cykly. Potom už jenom stačilo ukázat, že nemohou existovat cykly délky větší než 2. Objevila se i řešení, která místo nejbližších dvou politiků uvažovala nejdál hozené vejce. *(Josef Minařík)*

Problem 3.

In the plane, n points are coloured red in such a way that the distance between any two of them is at least 1. Prove that there are at most $3n$ pairs of red points whose distance is exactly 1. *(Lenka Kopfová)*

Solution:
Pick a red point A and suppose that there are k other red points with distance exactly 1 from A. All such points must lie on a unit circle centered at A, so they partition it into k arcs. Since the length of a circular arc connecting two points is strictly greater than their distance, each of these arcs must have length strictly bigger than 1, hence their total length (which is just the circumference of the circle) must be at least k. But the circumference of a unit circle is simply 2π, so we have $k < 2\pi$. Because k is an integer, this also means $k \leq 6$.

Therefore the total number of ordered pairs of red points (A, B) such that $|AB| = 1$ is at most $6n$, as for each of n choices for A we will have at most 6 choices for B. We are interested in the number of different unordered pairs, which is going to be equal to half of the number of ordered pairs by symmetry. Hence there are at most $\frac{1}{2} \cdot 6n = 3n$ pairs of red points with distance exactly 1, as we wanted to show.

Poznámky:
Většina řešení přistupovala k úloze podobným způsobem jako vzorák. Poměrně často se však objevovaly různé nepřesnosti ve zdůvodnění toho, proč může existovat nejvýše šest červených bodů s jednotkovou vzdáleností od červeného bodu. Nakonec jsem se ale rozhodl nestrhávat body, pokud se v řešení objevil aspoň nějaký nástin důkazu. Několik řešitelů se mě (neúspěšně) snažilo přesvědčit

o tom, že nalezli optimální konfiguraci. *Jaromír Potůček* si zasloužil imaginární bod za překvapivé využití teorie grafů: představíme-li si červené body jako vrcholy a jednotkové úsečky mezi nimi jako hrany, tak dostaneme rovinný graf s n vrcholy, na nějž můžeme použít Eulerovu formuli, abychom dostali silnější odhad, že může existovat nanejvýš $3n-6$ hran. *(Danil Koževnikov)*

Problem 4.

A quadrilateral $ABCD$ is inscribed in a circle ω with centre O in such a way that the diagonals AC and BD are perpendicular. Prove that the distance from O to the line CD is $\frac{1}{2}|AB|$.

(Radek Olšák)

Solution:

Let P be the intersection of the diagonals.

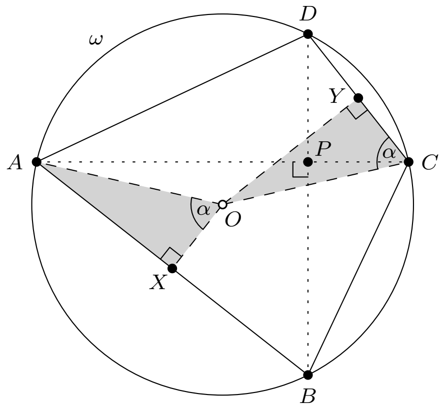
Firstly, we will show that O lies inside $ABCD$. For the sake of contradition, let O lie "beneath" AB. Then, by the inscribed angle theorem, we know that $\angle ACB > 90°$. Because P lies inside $ABCD$, $\angle APB > \angle ACB > 90°$, but $\angle APB = 90°$, which is a contradiction, hence O must lie inside the quadrilateral.

Let X and Y be the midpoints of AB and CD respectively. Because AB is an arc of ω, we have $OX \perp AB$. The same holds for CD and OY. Moreover, OY is the distance of O from CD.

We will show that the triangles AXO and OYC are congruent. Denote $\angle AOX = \alpha$, then $\angle AOB = 2\alpha$ and by the inscribed angle theorem $\angle ACB = \alpha$. Because the triangle CBP is right-angled, $\angle CBP = \angle CBD = 90° - \alpha$. Applying the inscribed angle theorem again, we get

$$\angle COD = 180^\circ - 2\alpha, \qquad \angle YOC = 90^\circ - \alpha \quad \text{and} \quad \angle YCO = \alpha = \angle AOX.$$

Hence AXO and OYC have two equal angles, so they are similar. Because $OA = OC$ is just the radius of ω, the triangles are congruent. Therefore $\frac{AB}{2} = AX = OY$, which is what we wanted to prove.



Poznámky:

Imagiraní bod jsem dala jediné řešitelce, kterou napadlo se zamyslet nad situací, kdy O je mimo $ABCD$. V takovém případě pak některé středové úhly nejsou rovny dvojnásobku, ale $180°$ mínus dvojnásobek obvodového úhlu, a důkaz by nemusel fungovat.

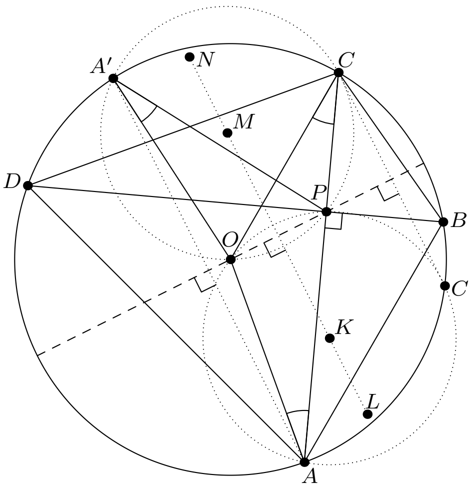
Až na tenhle zádrhel ale byla skoro všechna řešení správná. *(„madam Verča" Hladíková)*

Problem 5.

Let $ABCD$ be a cyclic quadrilateral. Let P denote the intersection of its diagonals and O its circumcentre. Finally, let K, L, M, N be the circumcentres of $\triangle AOP$, $\triangle BOP$, $\triangle COP$, $\triangle DOP$. Prove that $KL = MN$. *(Radek Olšák)*

SOLUTION:

Throughout the solution, all reflections will be across the line OP. We will show that K is the reflection of M and similarly that L is the reflection of N.



First note that AO and CO are the radii, so AOC is an isosceles triangle, hence $\angle OAP = \angle OCP$. Now let A' be the reflection of A. Since $\angle OCP = \angle OAP = \angle OA'P$, the quadrilateral $CPOA'$ is cyclic with M being its circumcentre. On the other hand, letting C' be the reflection of C, we get $\angle OAP = \angle OCP = \angle OC'P$, so $C'POA$ is a cyclic quadrilateral with K being its circumcentre. However, $CPOA'$ is the reflection of $C'POA$, so their circumcentres are also reflections of each other.

Analogously, if we reflect B to B' and D to D', we get cyclic quadrilaterals $B'OPD$ and $BOPD'$ that are reflections of each other, and so their circumcentres N and L are also reflections of each other. Therefore, the segments KL and MN are reflections of each other and thus have equal length.

POZNÁMKY:

Objevilo se mnoho různých postupů řešení, z nichž téměř všechny byly správně. Většina řešitelů si doplnila pár středů úseček na přímce OP a argumentovala pomocí shodností/podobnosti trojúhelníků. Ale objevily se například i sinová věta, spirální podobnost, mocnost bodu ke kružnici nebo izogonální kamarádi. Pokud jsem někde snižoval body, tak to bylo to za nejasné vysvětlení nebo chybějící kroky. *(Jáchym Solecký)*

Problem 6.

Given a regular 100-gon with 10 blue, 10 red, and 80 colourless vertices, prove that there must be a pair of blue vertices with the same distance as some pair of red vertices. *(Lenka Kopfová)*

Solution:

What does it mean for some pair of blue and red vertices to have the same distance? One way to look at it is that there exists a rotation of the 100-gon which maps two different blue vertices to some red vertices (if the red ones stay fixed).

We know that for each pair of a blue vertex and a red vertex there is exactly one nontrivial rotation which maps the blue one to the red one. Notice that we have exactly $10 \cdot 10 = 100$ such blue-red pairs.

There are 100 rotations of the 100-gon and one of them is the identity, so 99 nontrivial rotations. Hence we have 100 pairs and only 99 rotations which can take a blue element to the red element it is paired with. Therefore, by the pigeonhole principle, we get that at least one rotation must map at least two blue vertices to red vertices.

So there definitely exists a pair of blue vertices and a pair of red vertices with the same distance.

Poznámky:

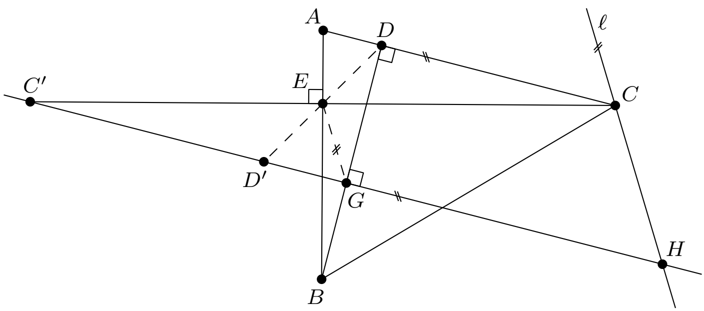
Vyskytly se dva typy řešení. Jedna skupina se vesměs ubírala stejnou či podobnou cestou jako vzorák. Těm byl udělen plný počet bodů. Poté nějaká řešení obsahovala náznak důkazu, a proto jim byly uděleny částečné body (většinou jeden). Celkově byla úloha trochu triková, proto zřejmě nepřišlo úplně moc řešení. *(Filip Čermák)*

Problem 7.

Let ABC be a triangle and BD, CE its altitudes. Let G be a point on BD such that $GE = DE$. Let ℓ denote the line through C parallel to GE. The line through G parallel to AC intersects ℓ at H. Prove that $BH = BC$. *(Radek Olšák)*

Solution:

Let C' and D' be points such that E is the midpoint of both CC' and DD'. It follows that $CDC'D'$ is a parallelogram, so $C'D' \parallel CD$. Because $ED = ED' = EG$, all three points D, D', G lie on a circle centered at E. This means that DD' is the diameter of this circle and thus $\angle DGD' = 90°$, so $D'G \perp GD \perp AC$, meaning $D'G \parallel AC$. In this way, we have shown that C', D', G and H all lie on a single line parallel to AC.



Now E is the midpoint of CC' and $EB \perp CC'$, so EB is the perpendicular bisector of CC' and thus $BC = BC'$. From $EG \parallel CH$, triangles $C'EG$ and $C'CH$ are similar, leading to

$$\frac{C'H}{C'G} = \frac{C'C}{C'E} = 2.$$

This means that G is the midpoint of $C'H$. Like before, $GB \perp C'H$, so GB is the perpendicular bisector of $C'H$ and thus $BC' = BH$. This proves that $BC = BC' = BH$.

Poznámky:

Většina došlých řešení obdržela plný počet bodů. Nejčastějším postupem bylo dokreslit bod S jako střed CH a vyúhlit $BS \perp SC$ přes tětivový pětiúhelník $BSCDE$. Mnohým řešitelům doporučuji důsledně uvádět, jak jsou nově zavedené body definovány, a nespoléhat se jen na obrázek – zaprvé by řešení mělo být úplné a srozumitelné i ze samotného textu, zadruhé z obrázku nelze poznat, které vlastnosti bod definují a které budou teprve dokázány. *(Matěj Doležálek)*

Problem 8.

Let S be a set of n points in the plane such that no four points lie in one line. Let $\{d_1, d_2, \dots, d_k\}$ be the set of distances between pairs of distinct points in S, and let m_i be the multiplicity of d_i, i.e. the number of unordered pairs of points whose distance is d_i. Prove that

$$m_1^2 + m_2^2 + \cdots + m_k^2 \le n^3 - n^2.$$

(Martin Raška)

Solution:

We will prove the desired inequality by double counting T, the number of isosceles triangles with vertices in S. More specifically, we will derive an upper and a lower bound on T yielding the desired inequality together. If there are any equilateral triangles in S, we will count them 3 times, sinces they can be viewed as 3 different isosceles triangles by choosing different sides as the base.

At first, let us get an upper bound on T. Choose two distinct points $A, B \in S$ and assume that AB is a base of some isosceles triangle. The apex of this triangle must lie on the perpendicular bisector of AB, therefore there are at most 3 different isosceles triangles with base AB, since no four points lie in one line. Altogether, $T \le 3\binom{n}{2}$, because we can choose $\binom{n}{2}$ different pairs of points as the base. Notice that equilateral triangles are counted 3 times in accordance with the first paragraph.

For the lower bound, we shall count the number of isosceles triangles whose legs have length d_i for a given i. For a fixed $A \in S$, denote $D(A,i)$ the number of points in S such that their distance to A is equal to d_i. Then there are exactly $\binom{D(A,i)}{2}$ isosceles triangles with apex A and legs of length d_i, so the total number of isosceles triangles with legs of length d_i is $\sum_{X\in S}\binom{D(X,i)}{2}$. It is clear that $\sum_{X\in S} D(X,i) = 2m_i$, as we count each pair of points with distance d_i exactly twice. Hence by Cauchy-Schwarz inequality in the form $n\left(\sum x_i^2\right) \ge \left(\sum x_i\right)^2$, we get

$$\sum_{X\in S}\binom{D(X,i)}{2} = \sum_{X\in S}\frac{D(X,i)^2 - D(X,i)}{2} = \frac{\sum_{X\in S} D(X,i)^2}{2} - \frac{2m_i}{2} \ge \frac{(2m_i)^2}{2n} - m_i = \frac{2m_i^2}{n} - m_i.$$

To summarize, there are at least $\frac{2m_i^2}{n} - m_i$ isosceles triangles with legs of length d_i. Since we want to count all the isosceles triangles, we have to sum this over all possible lengths. So

$$T = \sum_{i=1}^{k}\sum_{X\in S}\binom{D(X,i)}{2} \ge \sum_{i=1}^{k}\frac{2m_i^2}{n} - m_i = -\binom{n}{2} + \sum_{i=1}^{k}\frac{2m_i^2}{n},$$

where the last equality comes from the fact that $\sum_{i=1}^{k} m_i = \binom{n}{2}$, as there are $\binom{n}{2}$ pairs of points.

Using both the upper and the lower bound on T, we get

$$-\binom{n}{2} + \sum_{i=1}^{k}\frac{2m_i^2}{n} \le T \le 3\binom{n}{2},$$

which yields

$$\sum_{i=1}^{k} m_i^2 \le \frac{n}{2}\cdot 4\binom{n}{2} = 2n\frac{n(n-1)}{2} = n^3 - n^2.$$

Poznámky:
Všechna správná řešení postupovala vzorovou myšlenkou přes počítání dvojím způsobem, což je v kombinatorice obecně celkem častý a užitečný způsob, jak dokazovat (ne)rovnosti. Došlých řešení bylo bohužel poměrně málo. *(Martin Raška)*