Polynomials

4^{TH} AUTUMN SERIES — MODEL SOLUTIONS

Problem 1.

Daniel would like to have two polynomials $P(x)$, $Q(x)$ such that the degree of the product $P(x) \cdot Q(x)$ is six and the degree of the sum $P(x) + Q(x)$ is two. Help him find an example of such polynomials.

(Josef Minařík)

SOLUTION:
We can take $x^3 + x^2$ and $-x^3$ as an example of two polynomials satisfying the given conditions. Clearly, their sum is x^2 and their product is $-x^6 - x^5$.

POZNÁMKY:
Většina řešitelů našla dva polynomy. Taková řešení si vysloužila všechny tři body. Zde bych však do budoucna upozornil, že pokud napíšete něco navíc, je to super, ale dejte si pozor, ať je to správně.

(Vojta „Dláža“ Gaďurek)

Problem 2.

Klátra owns the polynomial $P(x) = x^2 + 8x + 12$. Prove that for any positive integer n the value $P(n)$ is not a prime number. *(Marian Poljak)*

SOLUTION:
We can easily factorize Klátra's polynomial $P(n)$ as

$$P(n) = n^2 + 8n + 12 = n(n + 6) + 2(n + 6) = (n + 2)(n + 6).$$

For any positive integer n both $n + 2$ and $n + 6$ are positive integers greater than 1. By definition, a prime number can only have two positive divisors (1 and itself). But $P(n)$ has at least two divisors ($n + 2$ and $n + 6$) greater than 1, so it can't be a prime number.

POZNÁMKY:
Všechna došlá řešení se úspěšně dobrala k cíli, často dost podobně jako to vzorové. Bohužel jsem ale některým řešitelům musela strhnout bod za sice zjevný, ale důležitý poznatek, že žádná ze závorek nemůže nabývat hodnoty jedna. *(Adéla Karolína „Áďa“ Žáčková)*

Problem 3.

Fíla gave Áďa three integers a, b, c for her birthday. Martin gave her a polynomial $P(x)$ with integer coefficients satisfying $P(a) = 1$, $P(b) = 2$ and $P(c) = 3$. Prove that b lies between a and c.

(Lenka Kopfová)

Solution:
We use a well-known fact from the introductory text that given integers a, b and a polynomial $P(x)$ with integer coefficients, we have $a-b \mid P(a)-P(b)$. By applying this fact to the problem, we get

$$\begin{aligned} b-a &\mid P(b)-P(a)=1, \\ c-b &\mid P(c)-P(b)=1. \end{aligned}$$

Therefore $(b-a),(c-b) \in \{1,-1\}$, which implies $a, c \in \{b-1, b+1\}$. But we know that $a \neq c$, as the polynomial P attains different values at a and c. So either $a=b+1$ and $c=b-1$, or $a=b-1$ and $c=b+1$. In both cases b lies between a and c, so we are done.

Poznámky:
Většina řešení byla správně, často ale postupovala trochu větším rozebíráním případů než vzorové řešení. Někteří řešitelé se snažili úlohu dokázat bez využití toho, že zadaná čísla i koeficienty polynomu jsou celá čísla. To ovšem nejde, protože pokud pro nějaký konečný počet bodů určíme funkční hodnoty v nich, vždy umíme najít polynom dostatečně velkého stupně, který v zadaných bodech určených funkčních hodnot nabývá. Celočíselnost tak v tomto případně byla naprosto klíčová, stejně jako zmíněné tvrzení, které bylo možno najít v úvodním textu této série. *(Lenka Kopfová)*

Problem 4.

Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with roots $x_1, \ldots, x_n$. Express $(x_1^2-1)\cdots(x_n^2-1)$ in terms of $a_0, a_1, \ldots, a_{n-1}$. *(Filip Čermák)*

Solution:
As we know from the introductory text, the polynomial $P(x)$ with roots $x_1, \ldots, x_n$ can be factorized as $a_n(x-x_1)(x-x_2)\cdots(x-x_n)$. Our polynomial $P(x)$ has a leading coefficient $a_n = 1$, therefore $P(x) = (x-x_1)(x-x_2)\cdots(x-x_n)$.

We want to express $(x_1^2-1)\cdots(x_n^2-1)$ in terms of coefficients of P, so we can rewrite it as

$$(x_1^2-1)\cdots(x_n^2-1) = (x_1-1)(x_1+1)\cdots(x_n-1)(x_n+1) = (x_1-1)\cdots(x_n-1)\cdot(x_1+1)\cdots(x_n+1),$$

using $a^2-b^2=(a-b)(a+b)$.

Now we can see that the final form of the expression is pretty similar to our factorization of $P(x)$. We can again rewrite the expression as

$$(x_1-1)\cdots(x_n-1)\cdot(x_1+1)\cdots(x_n+1) = (-1)^n(1-x_1)\cdots(1-x_n)\cdot(-1)^n(-1-x_1)\cdots(-1-x_n)$$

and since $(-1)^n(-1)^n = (-1)^{2n} = 1$, the powers of -1 just cancel out.

On the other hand, by evaluating $P(x)$ at ± 1, we get $P(1) = (1-x_1)\cdots(1-x_n)$ and $P(-1) = (-1-x_1)\cdots(-1-x_n)$. Plugging these expressions into $P(1)P(-1)$ and using the equalities we've derived above yields

$$P(1)P(-1) = (x_1^2-1)\cdots(x_n^2-1).$$

Finally, we need to express $P(1)P(-1)$ in terms of the coefficients of P to get the desired answer:

$$P(1)P(-1) = \Big(a_0 + a_1 + \cdots + a_{n-1} + 1\Big)\Big(a_0 - a_1 + \cdots + (-1)^{n-1}a_{n-1} + (-1)^n\Big).$$

Poznámky:
Většina řešení, která přišla, byla správně a postupovala podobně vzorovému řešení. Občas se vyskytla nějaká plus mínus jednička, za kterou byl stržen bod, ale to se stávalo jen sporadicky.
(Filip Čermák)

Problem 5.

Vašek hid his favourite n-tuple $a_1, \dots, a_n$ of real numbers in a vault and the code is

$$a_1^2 + a_2^2 + \cdots + a_n^2.$$

Majda would like to steal his n-tuple, but she only knows that it satisfies

$$1 + x^n + x^{2n} = (1 + a_1x + x^2) \cdot (1 + a_2x + x^2) \cdots (1 + a_nx + x^2)$$

for all real x. Help Majda find the code in terms of n. *(Marian Poljak)*

SOLUTION:

We will compare the coefficients of the polynomials from the equality in the problem statement. Let's first explore coefficients on the right-hand side. To obtain the linear term of the product, we must choose a_ix from one factor and 1 from the rest. So the coefficient of the linear term is $a_1 + \cdots + a_n$. We can get the quadratic term either by choosing a_ix and a_jx from two factors and 1 from the others, or simply by choosing x^2 from one factor and 1 from the rest. So the coefficient of the quadratic term is $n + \sum_{1 \le i < j \le n} a_ia_j$.

Now let's take a look at the left-hand side. If $n = 1$, the coefficients of the linear and the quadratic term are both 1. If $n = 2$, the coefficients of the linear and the quadratic term are 0 and 1, respectively. If $n > 2$, both these coefficients are 0.

If the two polynomials are equal for all real x, their coefficients must also be equal. Thus for $n = 1$, it is necessary that $a_1 = 1$. For $n = 2$, we get $a_1 + a_2 = 0$ and $a_1a_2 + 2 = 1$, so $a_1a_2 = -1$. For $n > 2$, we have $\sum_{i=1}^n a_i = 0$ and $n + \sum_{1 \le i < j \le n} a_ia_j = 0$, therefore $\sum_{1 \le i < j \le n} a_ia_j = -n$.

We know that

$$\sum_{i=1}^n a_i^2 = \left(\sum_{i=1}^n a_i \right)^2 - 2 \sum_{1 \le i < j \le n} a_ia_j.$$

On the left-hand side, we have the code we want to find. If we substitute the above values to the right-hand side, we obtain 1 for $n = 1$, then 2 for $n = 2$ and finally $2n$ for $n > 2$.

POZNÁMKY:

Většina řešení postupovala stejně jako vzorové, body jsem strhávala hlavně za nedořešení některých případů. *(Magdaléna Mišinová)*

Problem 6.

Find all non-constant polynomials P, Q with real coefficients which satisfy

$$P\left(Q(x)^3\right) = x \cdot P(x) \cdot Q(x)^3$$

for all real x. *(Matěj Doležálek)*

SOLUTION:

Both sides of the equation are polynomials, which are equal for all $x \in \mathbb{R}$ if and only if their coefficients are equal.

Let m and n be the degrees of P and Q, respectively. Then the equation implies

$$3mn = 1 + m + 3n.$$

By rearranging, we obtain

$$(3n - 1)(m - 1) = 2.$$

Since the degrees are positive integers, both factors must be positive integers. Therefore, there are only two options:

$$3n - 1 = 1 \ \& \ m - 1 = 2,$$
$$3n - 1 = 2 \ \& \ m - 1 = 1.$$

Due to $\frac{2}{3}$ not being an integer, the only solution is $n = 1$ and $m = 2$. Hence we can write

$$P(x) = p_2x^2 + p_1x + p_0,$$
$$Q(x) = q_1x + q_0$$

for some $p_2, p_1, p_0, q_1, q_0 \in \mathbb{R}$, such that $p_2, q_1 \neq 0$.

Let us compare the leading coefficients of both sides of the original equation: for the left-hand side we have $p_2 \cdot (q_1^3)^2$ and for the right-hand side we get $p_2 \cdot q_1^3$. Thus, $0 = p_2 \cdot q_1^6 - p_2 \cdot q_1^3 = p_2q_1^3 \cdot (q_1^3 - 1)$. Given $p_2, q_1 \neq 0$, it follows that $q_1 = 1$.

We'll substitute $x = -q_0$ into both sides. Since $Q(-q_0) = q_0 - q_0 = 0$, we have

$$P(0) = P\left(Q(-q_0)^3\right) = (-q_0) \cdot P(-q_0) \cdot Q(-q_0)^3 = 0,$$

so 0 is a root of P and hence $p_0 = 0$.

Let us now plug in the coefficients and simplify:

$$P\left(Q(x)^3\right) = (x + q_0)^3 \cdot \left(p_2(x + q_0)^3 + p_1\right),$$
$$x \cdot P(x) \cdot Q(x)^3 = x(p_2x^2 + p_1x) \cdot (x + q_0)^3.$$

Since the equality holds for $x \neq -q_0$, we have

$$p_2(x + q_0)^3 + p_1 = p_2x^3 + p_1x^2.$$

The linear coefficient on the left-hand side of this equation is $p_2 \cdot 3q_0^2$, but it must also be equal to 0, so $q_0 = 0$. The absolute coefficient then implies $p_1 = 0$.

Therefore, the only possible solutions are $P(x) = p_2x^2$ and $Q(x) = x$ for $p_2 \neq 0$. We can verify that all such pairs indeed satisfy the equation:

$$P\left(Q(x)^3\right) = p_2\left(x^3\right)^2 = p_2x^6,$$
$$x \cdot P(x) \cdot Q(x)^3 = x \cdot p_2x^2 \cdot x^3 = p_2x^6.$$

Solutions of the given equation are thus exactly the pairs $P(x) = p_2x^2$, $Q(x) = x$ for $p_2 \neq 0$.

Poznámky:

Většina řešitelů nalezla správná řešení úlohy, mnoho ale nepoužilo správné argumenty. Pokud je na pravé straně rovnice v součinu x, neznamená to nutně, že je absolutní člen $P(x)$ roven nule.[1] Je třeba vzít v potaz i koeficienty polynomu $Q(x)$ nebo třeba jako ve vzorovém řešení využít kořenu $Q(x)$. *(Hedvika Ranošová)*

[1]Protipříkladem budiž $P(x) = x + 1$, $Q(x) = x - 1$. Pak $P\left(Q(x)^3\right) = x^3 + 3x^2 + 3x = x(x^2 + 3x + 3)$, ale oba absolutní členy jsou nenulové.

Problem 7.

Tajl lost all his polynomials. He only knows that his polynomials were exactly those polynomials $P(n)$ with integer coefficients which satisfy

$$P(n) \mid n! + 2$$

for all positive integers n. Find all Tajl's polynomials. *(Matěj Doležálek)*

SOLUTION:

Let P be a polynomial satisfying the given condition. Note that P cannot have any positive integer root, i.e. $P(n) \neq 0$ for $n \in \mathbb{N}$. Let $n \in \mathbb{N}$ be arbitrary, then there obviously exists an $N \in \mathbb{N}$ satisfying $N \geq |P(n)|$ and $N \equiv n \mod P(n)$. The introductory text tells us that $a - b \mid P(a) - P(b)$ for all $a, b \in \mathbb{Z}$. In particular, this implies

$$P(n) \mid N - n \mid P(N) - P(n),$$

and hence

$$P(n) \mid P(N) \mid N! + 2.$$

However, $P(n)$ divides $N!$, so it must also divide 2.

It follows that $P(n) \in \{-2, -1, 1, 2\}$ for all $n \in \mathbb{N}$, which means $P(n)$ is bounded and thus has to be constant. The only numbers dividing both $1! + 2$ and $2! + 2$ are 1 and -1. It is easy to check that the constant polynomials $P(n) = 1$ and $P(n) = -1$ satisfy the given condition.

POZNÁMKY:

Téměř všechna správná řešení postupovala podobně jako to vzorové. Mnoho řešitelů uhodlo, že jedinými polynomy splňujícími zadání jsou $P(n) = 1$ a $P(n) = -1$, za což bohužel žádné body nebyly. Někteří také zapomněli na záporné řešení. *(Josef Minařík)*

Problem 8.

Ducky is swimming in a pond full of all polynomials. Polynomial $P(x)$ is called fishy if all of its coefficients are integers and there are infinitely many pairs (a, b) of coprime[2] positive integers, which satisfy

$$a + b \mid P(a) + P(b).$$

Find all fishy polynomials. *(Rado van Švarc)*

SOLUTION:

The zero polynomial is clearly fishy, so we may consider only non-zero polynomials from now on. Let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a non-zero polynomial of degree n with integer coefficients. We shall prove the following: P is not fishy if and only if there is exactly one even k, such that $0 \leq k \leq n$ and $c_k \neq 0$.

First, let us observe that for any positive integers a, b, we have $b \equiv -a \pmod{a + b}$. Since P has integer coefficients, this implies $P(a) + P(b) \equiv P(a) + P(-a) \pmod{a + b}$. Hence a polynomial is fishy precisely when there exist infinitely many pairs of coprime positive integers (a, b) such that $a + b \mid P(a) + P(-a)$. We can also note that, by Euclid's algorithm, we have $\gcd(a + b, a) = \gcd(a, b) = 1$.

Note that in the expression $P(a) + P(-a)$, all the odd powers of a end up cancelling out, thus the fishiness of P is not affected by any particular choice of coefficients c_ℓ, where ℓ is a positive odd integer. Therefore we can, without loss of generality, assume that $c_\ell = 0$ for all odd ℓ, so that P is an even polynomial, i.e. of the form $P(x) = c_{2m}x^{2m} + c_{2m-2}x^{2m-2} + \cdots + c_0$ for a non-negative integer m and non-zero c_{2m}.

[2] Two numbers are called coprime if their greatest common divisor is one.

If c_{2m} is the only non-zero coefficient of P, then all the divisors of $P(a) + P(-a) = 2c_{2m}a^{2m}$ that are relatively prime to a must divide $2c_{2m}$. In particular, they all must be less than or equal to $2c_{2m}$, so only pairs with $a+b \leq 2c_{2m}$ can satisfy the divisibility and our polynomial is not fishy.

On the other hand suppose that $c_{2k} \neq 0$ for some $0 \leq k < m$, and pick the smallest such k. Further, let a be a prime and $d = |c_{2k}a^{2m-2k} + c_{2k-2}a^{2m-2k-2} + \cdots + c_{2k}| = \left|\frac{P(a)}{a^{2k}}\right|$. Clearly, d is a divisor of $2P(a) = P(a) + P(-a)$. By Euclid's algorithm, if $a > |c_{2k}|$, we have

$$\gcd(d, a) = \gcd(c_{2k}a^{2m-2k} + c_{2k-2}a^{2m-2k-2} + \cdots + c_{2k}, a) = \gcd(c_{2k}, a) = 1.$$

Also, since $2m - 2k \geq 2$ and d is defined as the absolute value of a polynomial in a with degree $2m - 2k$, we will have $d > a$ for all large enough numbers a. Hence for any sufficiently large prime a, this choice of d provides us with a pair $(a, b) = (a, d - a)$ that satisfies all the hypotheses of the problem. Since there are infinitely many primes, this proves that all such polynomials are fishy.

To sum up, we have shown that if P has exactly one non-zero even-indexed coefficient, it is not fishy, and that otherwise, it is fishy, which is precisely what we wanted.

Poznámky:

Někteří řešitelé pouze poznamenali, že všechny liché polynomy vyhovují zadání (nebo se mě o tom dokonce pokusili více či méně seriózně přesvědčit), za což jsem žádné body nedával. Zbylá řešení mě ovšem velmi potěšila a strhával jsem body pouze za všemožné nepřesnosti jako například opomenutí degenerovaného případu, kdy má P pouze jeden nenulový sudý koeficient. *(Danil Koževnikov)*