Arrangements

4. podzimní série — Vzorové řešení

Úloha 1.

There are n pigs standing in a line. Among those, however, Matouš, Matěj and Michal do not want to stand next to each other[1]*. Find the number of possible ways to arrange the pigs that satisfy this condition.*

Řešení:

There are $n-3$ pigs whose placement in the line is not constrained in any way. Therefore, there are $(n-3)!$ possible arrangements of these pigs. Then there are $n-2$ slots where Matěj, Michal, and Matouš may be inserted ($n-4$ slots between some other two pigs and 2 slots at the ends of the line). Since they do not want to stand next to each other, only one of them can be put in any slot. Thus, we need to choose 3 out of these $n-2$ slots, and then the remaining three pigs can be arranged in 3! ways in these 3 chosen slots. So in total, the number of satisfactory ways of arranging the pigs is $(n-3)!\binom{n-2}{3}3!$.

Poznámky:

Nemalá část řešitelů postupovala podobně jako vzorové řešení a přímo spočítala validní uspořádání prasátek. Někteří zvolili opačný postup a od všech možných uspořádání odčítali ta, která nevyhovují zadání, jelikož jsou v nich nějaká prasátka od M vedle sebe. Takové řešení ale skrývá nemalé úskalí v překryvu počítaných možností podle jednotlivých prasátek od M, se kterým se zvládla vypořádat jen část řešitelů. *(Klárka Grinerová)*

Úloha 2.

Sylva found a clock that had its numbers rearranged. For each of the twelve neighbouring pairs of numbers, she wrote down their sum. She then replaced each of these twelve sums with its remainder when divided by 13. Finally, she summed the twelve remainders. What is the smallest value Sylva could have obtained out of all possible arrangements?

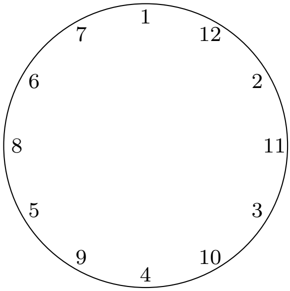
Řešení:

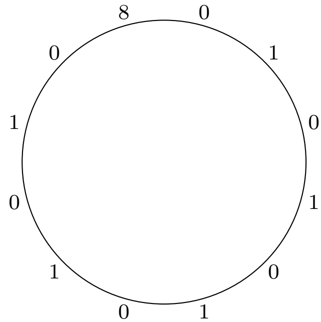
We start by showing that any obtained value v must be a multiple of 13. It is important to remember the fact that $(a \bmod n + b \bmod n) \bmod n = (a+b) \bmod n$ when n is prime. We add all remainders together modulo 13. Using the aforementioned fact, what matters is how many times each number appears in the remainders. Each number appears in exactly two different remainders. Thus, we get $2\sum_{i=1}^{12} i \bmod 13$ which is equal to 0. This means that every v is a multiple of 13.

Now, let us continue by showing that it cannot be 0. Consider 1. The only number satisfying $1 + a \bmod 13 = 0$ is $a = 12$. However, for v to be 0, every remainder in the sum must be 0. But 1 is

[1] Matouš, Matěj and Michal are pigs.

part of two remainders. Lastly, we show that there exists an arrangement where $v = 13$. One such arrangement is $1, 12, 2, 11, 3, 10, 4, 9, 5, 8, 6, 7$.





Therefore 13 is the best answer.

Úloha 3.

Štepi is playing with a grid of 2024×2024 points. For every ordered triplet of distinct points (A, B, C) of the grid, he measures[2] *and writes down $\angle ABC$. What is the average of all the numbers Štepi writes down?*

Řešení:

Firstly, note that we are looking for an *average*. Let us denote by S the sum of all the angles and by n their count – then the desired answer is $\frac{S}{n}$.

Let us note that we are considering all the angles corresponding to *ordered* triplets of distinct points in the grid. We will divide all of these triplets into groups of 6, with each group consisting of the six permutations of some three points A, B, C. In other words, the angles $\sphericalangle ABC$, $\sphericalangle BCA$, $\sphericalangle CAB$, $\sphericalangle ACB$, $\sphericalangle CBA$ and $\sphericalangle BAC$ form one group. Since the angles in a triangle always add up to $180°$, it holds that $\sphericalangle ABC + \sphericalangle BCA + \sphericalangle CAB = 180°$ and analogously $\sphericalangle ACB + \sphericalangle CBA + \sphericalangle BAC = 180°$. Therefore, the sum of angles in one group will be $180° \cdot 2 = 360°$. Since we have exactly n angles contributing to S, there will be $\frac{n}{6}$ groups in total. This means that S will equal $\frac{n}{6} \cdot 360°$.

Now we just perform the following calculation to obtain:

$$\frac{S}{n} = S \cdot \frac{1}{n} = \left(\frac{n}{6} \cdot 360°\right) \cdot \frac{1}{n} = (n \cdot 60°) \cdot \frac{1}{n} = 60°.$$

With this trick, we have easily found the average to be $60°$.

Poznámky:

Téměř všechna řešení byla správná a použila tentýž trik jako vzorák. *(Lenka Poljaková)*

[2]Of the two angles determined by rays BA and BC, Štepi always measures the smaller one.

Úloha 4.

Let n be a positive integer such that its base-10 representation contains each of the digits 0, 1, 2 and 3 at least once. Show that the digits of n can be permuted so that the new number[3] is divisible by 7.

Řešení:
Let us remove one of each of the digits 0, 1, 2, 3 from n and label the resulting number m. We will then find a number n' divisible by 7 in the form $n' = 10000m + p$, where p will be some number formed by a permutation of the digits 0, 1, 2, 3. Note that the digits of n can indeed be permuted to get such an n'.

Let $r \in \{0, 1, 2, 3, 4, 5, 6\}$ be the remainder of $10000m$ modulo 7. We determine p based on r. For that, let us choose p as one of the numbers 0231, 1023, 2130, 0213, 0123, 1230, 0132. The remainders of these numbers modulo 7 are 0, 1, 2, 3, 4, 5, 6 respectively. For $r > 0$, let us select p so that its remainder is $7 - r$, and for $r = 0$, let us select p so that its remainder is 0.

For $r = 0$, we have

$$n' \equiv 10000m + p \equiv 0 + 0 \equiv 0 \pmod 7,$$

whereas for $r > 0$ we have

$$n' \equiv 10000m + p \equiv r + (7 - r) \equiv 0 \pmod 7.$$

Therefore, n' is divisible by 7 in either case. As we mentioned before, the digits of n can be permuted to form n', hence n' is the new number we were looking for.

Poznámky:
Väčšina riešení mala veľmi podobný postup ako vzorové riešenie. *(Michal „Miško“ Pecho)*

Úloha 5.

Vítek owns a deck of 2024 cards, each of which has one of four suites. Initially, the deck is sorted in such a way that any four consecutive cards are of four different suites. Vítek then takes some consecutive block of cards from the top of the deck, reverses its order and inserts it back somewhere into the deck. Afterwards, Vítek separates the deck into quadruplets, consisting of the first through fourth card, then fifth through eighth, etc. Show that each of these quadruplets contains cards of four distinct suites.

Řešení:
We can represent different suits with numbers from one to four. When any four consecutive cards are from different suits, the fifth card will be the same as first, the sixth the same as second and so on. So we know that the original order of cards was WLOG 1, 2, 3, 4, 1, 2, 3, 4, 1, ... Then after Vítek makes the changes, we can separate the deck into three parts: the unchanged beginning, the reversed part and the unchanged end (if the cards are put in the beginning or in the end, we are left with just two parts).

First, we take quadruplets from the beginning until there are k leftover cards, where $k < 4$. We know that these quadruplets are each from different suit, because they are from the unchanged part of the deck. Then we do the same thing from the end of the deck (for the same reason there will be ℓ cards remaining, where $\ell < 4$).

If we continue from the end, there are these ℓ cards, which will be the first ℓ cards from the sequence 4, 3, 2, 1. Then, if there are enough reversed cards, we take first $4 - \ell$ cards from them. These cards are from the sequence 1, 2, 3, 4, because they are the first cards from the unchanged deck but reversed. Therefore, these $\ell + (4 - \ell) = 4$ cards are each from a different suit.

[3] We allow the new number to begin with zeroes.

If there are not enough cards in the reversed part of deck to do this, there must be only four cards remaining in total (because $k < 4$ and there are less than four other cards) and because number of cards from each suit is the same and we have removed an equal amount from each suit, they must each be from different suit.

After that, we continue from the end and make quadruplets from the reversed cards until there are only m of them, where $m < 4$. Now we have only four cards remaining (because $m + k < 8$), and so the argument is the same as in the last paragraph. And with that we have proven, that each quadruplet always contains cards from all suites.

POZNÁMKY:
Protože nevíme, kam do balíčku Vítek karty vložil, tak musíme pokrýt všechny možné případy. Někteří řešitelé to ze zadání nepochopili a vkládali karty pouze na některá specifická místa. Zbylá řešení ale byla téměř všechna správná.

(Lukáš Trojan)

Úloha 6.

Majda and Vašek are playing a game, in which Majda takes the first turn and then they alternate. Initially, the numbers 2000, 1999, ..., 3, 2, 1 are written on a board in this order. During his turn, Vašek can choose 1000 numbers and rearrange them as he wishes. Majda can, during her turn, choose k numbers and rearrange them, where k is a fixed positive integer. Majda wins if the numbers on the board are in the order 1, 2, 3, ..., 1999, 2000. What is the smallest k for which Majda can always win (regardless of how Vašek plays) after a finite number of turns?

ŘEŠENÍ:
Let us observe: if one number is not in its place, at least one other number has to be in a wrong spot. (If some k is at the position $i \neq k$, then i also has to be in the wrong position.)

First of all, we prove that if $k < 1002$, then Majda cannot win. If at the start of Majda's turn more than k numbers are at wrong positions, then she cannot get all of them to their correct positions in one turn. Because of that, there are at least two numbers in wrong positions at the end of her turn (this derives from the observation at the start). Therefore, Vašek can shuffle some 1000 numbers, so that after his turn there are at least 1002 numbers in a wrong spot. Thus we proved that k has to be at least 1002.

Now we will present an algorithm for Majda that results in her winning the game when $k = 1002$. The algorithm is simple: take the 1000 numbers which Vašek has shuffled, return them wherever they were before Vašek shuffled them, and with the last two moves, pick a number not in its place (if such a number exists) and move it to its place. Thus the number of correctly placed numbers after her turn increases (compared to the aftermath of her previous turn) by at least one.

With this algorithm and $k = 1002$, Majda can win in at most 2000 turns.

POZNÁMKY:
Většinu řešení bylo možné rozdělit do dvou skupin. První měla pro $k \neq 1001$ stejný argument jako vzorové řešení. Druhá si čísla rozdělila na menší rovno 1000 a vetší než 1000. Následně ukázali, že Vašek vždy dokáže ve svém tahu navrátit všechna menší čísla na pozice od 1001 dál a obráceně.

(Petr Hladík)

Úloha 7.

Let p be an odd prime number and S_p be the set of permutations of the set $\{1, 2, \ldots, p\}$. For any $\pi \in S_p$, define $\Phi(\pi)$ as the number of multiples of p among the numbers

$$\pi(1), \qquad \pi(1) + \pi(2), \qquad \ldots, \qquad \pi(1) + \pi(2) + \cdots + \pi(p).$$

Find the value of $\displaystyle \frac{1}{p!} \sum_{\pi \in S_p} \Phi(\pi)$.

ŘEŠENÍ:

Let us first define a permutation shift. If we shift the permutation $\pi \in S_n$ transforms into $k \in \mathbb{Z}$, we denote the new permutation $\pi_{+k} \in S_n$ and define it as

$$\pi_{+k}(i) = \pi(i) + k \pmod{p} \qquad k \in \mathbb{Z}, i \in \{1, 2, \ldots, p\}.$$

For example, permutation $(1, 2, 3)$ shifted by 2 is $(1, 2, 3)_{+2} = (3, 1, 2)$. It is pretty easy to see, that $(\pi_{+k})_{+l} = \pi_{+(k+l)}$ i.e. by shifting a permutation multiple times, we only get the same permutation shifted by a different number. And since $\pi_{+p} = \pi_{+0} = \pi$, shifting one permutation by various numbers can give us only p different permutations. These permutations $\pi_{+0}, \pi_{+1}, \ldots, \pi_{+(p-1)}$ are all different, because they all differ in the first element $\pi(1)$.

That said, it is obvious that the set of permutations S_n can be divided into p-element subsets, where each subset contains only permutations that are mutually shifted. Let us take one such subset $G \subset S_n$ and focus on computing the sum $\sum_{\pi \in G} \Phi(\pi)$. Let us define

$$s_n(\pi) = \sum_{i=1}^{n} \pi(i) = \pi(1) + \pi(2) + \ldots + \pi(n)$$

and an indicator function $I_n(\pi) = 1$ if p divides $s_n(\pi)$ and 0 otherwise. Then $\Phi(\pi)$ can be written as a sum of these indicators. Then the order of summation can be changed. We know from earlier, that our subset G can be written as $G = \{\pi_{+k}; k \in \mathbb{Z}\} = \left\{\pi_{+0}, \pi_{+1}, \ldots, \pi_{+(p-1)}\right\}$, where $\pi \in G$ is one of its permutations. This can be used to express the second sum

$$\sum_{\pi \in G} \Phi(\pi) = \sum_{\pi \in G} \sum_{n=1}^{p} I_n(\pi) = \sum_{n=1}^{p} \sum_{\pi \in G} I_n(\pi) = \sum_{n=1}^{p} \sum_{k=0}^{p-1} I_n(\pi_{+k}).$$

Proposition. *For $0 < n < p$, there is exactly one number among $s_n(\pi_{+0}), s_n(\pi_{+1}), \ldots, s_n\left(\pi_{+(p-1)}\right)$ divisible by p. Or equivalently for every $0 < n < p$ it holds that $\sum_{k=0}^{p-1} I_n(\pi_{+k}) = 1$.*

Proof. Let $A \equiv s_n(\pi) \pmod{p}$. Then we can compute

$$s_n(\pi_{+k}) = \sum_{i=1}^{n} \pi_{+k}(i) \equiv \sum_{i=1}^{n} (\pi(i) + k) = s_n(\pi) + nk \equiv A + nk \pmod{p}.$$

It is known and can be easily proven, that since n and p are coprime, each of the numbers nk has different remainder modulo p. Thus also each of the numbers $A + nk = s_n(\pi_{+k})$ has different remainder modulo p for every $k \in \{0, 1, \ldots, p-1\}$. And since there are p such numbers and p possible remainders, exactly one of these numbers $s_n(\pi_{+k})$ has remainder 0 and is divisible by p. $\square$

We proved that the sum $\sum_{k=0}^{p-1} I_n(\pi_{+k}) = 1$ for every $n < p$. Let us now solve the case of $n = p$. The number $s_p(\pi) = \sum_{i=1}^{p} \pi(i)$ is actually a sum of all the numbers from 1 to p. Only their order is different for different permutations

$$s_p(\pi) = \sum_{i=1}^{p} \pi(i) = \sum_{i=1}^{p} i = \frac{p(p+1)}{2}.$$

Because p is odd, $\frac{p+1}{2}$ is an integer and $s_p(\pi)$ is an integer multiple of p. So $I_p(\pi) = 1$ for every $\pi \in S_n$. We can now compute the sum

$$\sum_{\pi \in G} \Phi(\pi) = \sum_{n=1}^{p} \sum_{k=0}^{p-1} I_n(\pi_{+k}) = \sum_{n=1}^{p-1} 1 + \sum_{k=0}^{p-1} I_p(\pi_{+k}) = p - 1 + p = 2p - 1.$$

At the beginning, we divided the permutations into subsets of size p. There are $p!$ of all the permutations, so the number of groups needs to be $\frac{p!}{p}$. Since the sum of $\Phi(\pi)$ in each group is $2p-1$, we can compute the final result.

$$\frac{1}{p!}\sum_{\pi\in S_p}\Phi(\pi)=\frac{1}{p!}\sum_{G\subset S_p}\sum_{\pi\in G}\Phi(\pi)=\frac{1}{p!}\sum_{G\subset S_p}(2p-1)=\frac{1}{p!}\frac{p!}{p}(2p-1)=\frac{2p-1}{p}.$$

POZNÁMKY:
Téměř všechna odevzdaná řešení byla správně. *(Ondra Trinkewitz)*

Úloha 8.

Matěj and Daník are standing in (not neccessarily the same) vertices of the complete graph on n vertices.[4] *Each edge of this graph has a price, which is a nonnegative real number that has to be paid when moving along this edge, and each price is unique. Both Daník and Matěj make a journey that visits each vertex exactly once, according to the following rules:*

(1) *Daník likes expensive things, so at each step, he moves along the edge which costs the most among the ones leading to vertices he hasn't visited yet.*
(2) *Matěj likes cheap things, so at each step, he moves along the edge which costs the least among the ones leading to vertices he hasn't visited yet.*

Show that in the end, Daník's total expenses are greater than or equal to Matěj's expenses.

SOLUTION INSPIRED BY VERČA MENŠÍKOVÁ:
Let us (without loss of generality) name the vertices of our complete graph 1, 2, …, n so that in each step Daník, who starts at vertex 1, moves from vertex i to vertex $i+1$. Matěj's journey then follows these vertices in some order. By $\{u,v\}$ we will mean the edge between the vertices u and v. Furthermore, we will denote the price of edge $\{u,v\}$ by $p(u,v)$.

We want to pair Daník's and Matěj's edges so that for each pair it holds that Daník's edge is more (or equally) expensive as Matěj's edge.

We will pair Matěj's edges in order of his journey in the following way: Let us have an edge in Matěj's journey, going from vertex m to some other vertex. We will pair it with Daník's edge $\{d,d+1\}$ where d is greatest possible such that $d\le m$ and the edge $\{d,d+1\}$ has not been paired yet.

Lemma. (which will be useful later in the solution) *Consider that we want to pair Matěj's edge going from vertex m. If we have already used the edge $\{m,m+1\}$ in our pairing, we must have also used all the edges $\{m+1,m+2\}$, $\{m+2,m+3\}$, …, $\{n-1,n\}$.*

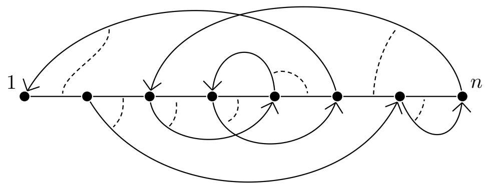
Proof. We will proceed by induction. First, let's remark that for $m=n-1$ we have already used the edge $\{n-1,n\}$ by the assumption of the lemma. Thus the base case of induction is done.

Now, let us suppose that the lemma holds for all $k>m$. If we have already used the edge $\{m,m+1\}$, there must exist a vertex $v>m$, with whose edge we have paired that edge. By the definition of the pairing, we know that the edges $\{m+1,m+2\},\dots,\{v-1,v\},\{v,v+1\}$ must have been paired before pairing the edge from v. By the inductive hypothesis we then know that also the edges $\{v+1,v+2\},\dots,\{n-1,n\}$ have been already paired. Therefore all edges $\{m+1,m+2\},\dots,\{n-1,n\}$ have been already paired before pairing the edge of v, which concludes the proof. □

Let us now show that our pairing is indeed feasible. We will prove this by contradiction. Assume we have Matěj's edge $\{u,v\}$ which can't be paired with any of Daník's edges. Then we know that the

[4] To learn what a *graph* or a *complete graph* is, see this older introductory text: *https://prase.cz/archive/42/uvod4p.pdf.*

edge $\{u, u+1\}$ has already been paired and by the lemma all the edges $\{u+1, u+2\}, \ldots, \{n-1, n\}$ too. And because we can't pair Matěj's edge $\{u, v\}$, all the edges $\{1, 2\}, \ldots, \{u-1, u\}$ have been paired too. Therefore we can see that all the edges have already been paired. We can't pair the same Matěj's edge twice and the amount of Matěj's and Daník's edges is the same. Thus we must have used all of Matěj's edges, which is a contradiction with the edge $\{u, v\}$ being unpaired.



Now it suffices to prove that for each pair Daník's edge is at least as expensive as Matěj's is. Let us have Matěj's edge $\{u, v\}$. We will divide the possible pairings into two cases:

(1) The edge was paired with the edge $\{u, u+1\}$. Because there are $n-u$ vertices above u and only $n-u-1$ edges above the edge $\{u, u+1\}$, there must exist a vertex w (not necessarily different from v) such that $w > u$, which Matěj has not visited yet. Otherwise, Matěj would have paired all the edges above $\{u-1, u\}$ with those vertices. So Matěj had the opportunity to choose the edge $\{u, w\}$. Because $w \geq u$, Daník must have had the opportunity to choose the edge $\{u, w\}$. That said and remembering that Matěj chooses the cheapest edge and Daník chooses the most expensive edge, we have the following inequality:
$$p(u, v) \leq p(u, w) \leq p(u, u+1).$$

(2) The edge was paired with the edge $\{x, x+1\}$ for some $x < u$. Therefore Matěj could not have visited the vertex x yet (otherwise he would have already used this edge). So in vertex u, Matěj had the opportunity to choose the vertex x as his next (here x is also not necessarily different from v). Also, Daník must have had the opportunity to choose the vertex u instead of $x+1$, when standing in x. From that we similarly get:
$$p(u, v) \leq p(u, x) = p(x, u) \leq p(x, x+1).$$

We now know that we can pair each Matěj's edge $\{u, v\}$ with some Daník's edge in such a way that the Matěj's edge is at most as expensive as Daník's. Now if we sum it over all edges, we have that Matěj's total expenses are at most as great as Daník's, which proves our problem.

Poznámky:

Z jedenácti statečných, kteří se s úlohou odhodlali prát, ji několik úspěšně vyřešilo. Mnohdy používali jiný postup než ten ve vzorovém řešení, kdy se snažili dokázat, že pro libovolné x má Daník víc hran dražších než toto x, než jich má Matěj. Bohužel, někteří řešitelé ne úplně řešili to, že některé hrany mohou být nepoužitelné (protože cestující ve vrcholech, kam vedou, už byl), nebo se snažili dokázat nepravdivé tvrzení, že v každém kroku vybere Matěj levnější hranu než Daník. Těm jsem bohužel nemohla dát ani částečné body. Nakonec bych jenom chtěla poopravit časté chyby: vrchol - vertex, vrcholy - vertices. *(Adéla Karolína „Áďa“ Žáčková)*