Remainders

4^{TH} AUTUMN SERIES DATE DUE: 5^{TH} JANUARY 2026

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
In how many ways can we arrange the numbers 1, 2, 3, 4, 5, 6 and 7 in a row so that the sum of the first n numbers is not divisible by 3 for any $n \in \{1, 2, \ldots, 7\}$?

PROBLEM 2. (3 POINTS)
There are $n \geq 2$ pairwise distinct positive integers arranged in a circle. For each number, Káťa writes down its remainder when divided by its neighbour in the clockwise direction. Is it possible that all the numbers she wrote down are equal?

PROBLEM 3. (3 POINTS)
Consider a sequence $a_1, a_2, \ldots, a_{2025}$ where each term is equal to either 2 or 3. Suppose there exists an integer sequence $b_1, b_2, \ldots, b_{2025}$ satisfying

$$a_i b_i + b_{i+2} \equiv 0 \pmod 5$$

for every $i = 1, 2, \ldots, 2025$ (interpret b_{2026} as b_1 and b_{2027} as b_2). Show that all the numbers $b_1, b_2, \ldots, b_{2025}$ are divisible by 5.

PROBLEM 4. (5 POINTS)
Find all pairs of integers (a, b) such that $a^{2026} + (a+b)^{2026} + b^{2026}$ is a perfect square.

PROBLEM 5. (5 POINTS)
A school in PraSe district has grades 1 through 12 with 100 students in each grade. As such, there are 1200 lockers (numbered 1 through 1200), one for each student. One day, all the students calculated the remainder their locker number gives when divided by their grade number and compared the results. They would call a grade *great* if all the students in that grade got the same remainder. Is it possible that the lockers were assigned in a way so that all 12 grades were great?

PROBLEM 6. (5 POINTS)
The numbers $1, 2, \ldots, 2n$ are divided into two piles of size n. For each of the n^2 ordered pairs in the first pile, Patrik takes the sum of the numbers of that pair, divides it by $2n$ and writes down the resulting remainder on a page in his notebook. He then does the same with pairs from the second pile, writing the results on another page. Prove that both pages contain each remainder the same number of times.

PROBLEM 7. (5 POINTS)
Show that the numbers $1, 3^3, 5^5, \ldots, (2n-1)^{2n-1}$ give pairwise distinct remainders when divided by 2^n.

PROBLEM 8. (5 POINTS)

Let $p > 3$ be a prime such that the smallest positive integer s satisfying $p \mid 2^s - 1$ is $s = p - 1$. Let $k = \frac{p-3}{2}$. We define a sequence $(a_n)_{n=1}^{\infty}$ by

$$a_i = a_{i+k} = 2^i \quad (1 \le i \le k), \qquad a_{j+2k} = a_j a_{j+k} \quad (j \ge 1).$$

Show that there exist $2k$ consecutive terms $a_{x+1}, a_{x+2}, \ldots, a_{x+2k}$ which are pairwise distinct modulo p.