Remainders

4TH AUTUMN SERIES — DATE DUE: 5TH JANUARY 2026

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
In how many ways can we arrange the numbers 1, 2, 3, 4, 5, 6 and 7 in a row so that the sum of the first n numbers is not divisible by 3 for any $n \in \{1, 2, \ldots, 7\}$?

PROBLEM 2. (3 POINTS)
There are $n \geq 2$ pairwise distinct positive integers arranged in a circle. For each number, Káťa writes down its remainder when divided by its neighbour in the clockwise direction. Is it possible that all the numbers she wrote down are equal?

PROBLEM 3. (3 POINTS)
Consider a sequence $a_1, a_2, \ldots, a_{2025}$ where each term is equal to either 2 or 3. Suppose there exists an integer sequence $b_1, b_2, \ldots, b_{2025}$ satisfying

$$a_i b_i + b_{i+2} \equiv 0 \pmod 5$$

for every $i = 1, 2, \ldots, 2025$ (interpret b_{2026} as b_1 and b_{2027} as b_2). Show that all the numbers $b_1, b_2, \ldots, b_{2025}$ are divisible by 5.

PROBLEM 4. (5 POINTS)
Find all pairs of integers (a, b) such that $a^{2026} + (a+b)^{2026} + b^{2026}$ is a perfect square.

PROBLEM 5. (5 POINTS)
A school in PraSe district has grades 1 through 12 with 100 students in each grade. As such, there are 1200 lockers (numbered 1 through 1200), one for each student. One day, all the students calculated the remainder their locker number gives when divided by their grade number and compared the results. They would call a grade *great* if all the students in that grade got the same remainder. Is it possible that the lockers were assigned in a way so that all 12 grades were great?

PROBLEM 6. (5 POINTS)
The numbers $1, 2, \ldots, 2n$ are divided into two piles of size n. For each of the n^2 ordered pairs in the first pile, Patrik takes the sum of the numbers of that pair, divides it by $2n$ and writes down the resulting remainder on a page in his notebook. He then does the same with pairs from the second pile, writing the results on another page. Prove that both pages contain each remainder the same number of times.

PROBLEM 7. (5 POINTS)
Show that the numbers $1, 3^3, 5^5, \ldots, (2n-1)^{2n-1}$ give pairwise distinct remainders when divided by 2^n.

PROBLEM 8. (5 POINTS)

Let $p > 3$ be a prime such that the smallest positive integer s satisfying $p \mid 2^s - 1$ is $s = p - 1$. Let $k = \frac{p-3}{2}$. We define a sequence $(a_n)_{n=1}^{\infty}$ by

$$a_i = a_{i+k} = 2^i \quad (1 \leq i \leq k), \qquad a_{j+2k} = a_j a_{j+k} \quad (j \geq 1).$$

Show that there exist $2k$ consecutive terms $a_{x+1}, a_{x+2}, \ldots, a_{x+2k}$ which are pairwise distinct modulo p.

Remainders

4. PODZIMNÍ SERIES — MODEL SOLUTIONS

Problem 1.

In how many ways can we arrange the numbers 1, 2, 3, 4, 5, 6 *and* 7 *in a row so that the sum of the first* n *numbers is not divisible by* 3 *for any* $n \in \{1, 2, \ldots, 7\}$*?*

SOLUTION:

We will denote by ♠ the condition that the sum of the first n numbers is not divisible by 3. We will divide the numbers 1 to 7 into three sets according to their remainders.

- 1, 4, 7 – We will call this set "ones" and each of its elements "one".
- 2, 5 – We will call this set "twos" and each of its elements "two".
- 3, 6 – We will call this set "zeros" and each of its elements "zero".

We notice that for each rearrangement A of numbers $1, 4, 7, 2, 5$ fulfilling ♠, we can obtain $5 \cdot 6 = 30$ rearrangements of numbers $1, \ldots, 7$ fulfilling ♠. That can be done by placing two "zeros" (3 and 6) in positions 2 to 7 and by placing the numbers from A in the remaining positions from left to right. We notice that each rearrangement of $1, \ldots, 7$ is obtained exactly once because ♠ implies that the first number is non-zero and inserting zeros anywhere except the first position does not violate the validity of the condition.

There are $2!\binom{6}{2} = 5 \cdot 6 = 30$ possibilities for how to place them because we have 6 positions for the first one and 5 positions for the second one. So there are 30 times more rearrangements of $1, \ldots, 7$ fulfilling ♠ than rearrangements of $1, 4, 7, 2, 5$ fulfilling ♠.

Therefore, it is enough to count the number of rearrangements of twos and ones in $(1, 4, 7, 2, 5)$ fulfilling ♠. We notice that remainders modulo 3 in each rearrangement have to be $1, 1, 2, 1, 2$ in this order.

To prove it we will consider two cases: To fulfil ♠, if the rearrangement starts with "one", second number must be "one", third "two", fourth "one", fifth "two". If it starts with "two", second number must be "two", third "one" and fourth "two", which is a contradiction as there are only two "twos". Conversely, each rearrangement "of the type 1 1 2 1 2" fulfils ♠. There are $3! \cdot 2!$ such rearrangements because "ones" can be rearranged in 2! ways and "twos" in 3! ways.

The answer is $3! \cdot 2! \cdot 30 = 360$.

POZNÁMKY:

Většina správných řešení postupovala podobně jako vzorové řešení. *(Ivan Žemlička)*

Problem 2.

There are $n \geq 2$ *pairwise distinct positive integers arranged in a circle. For each number, Káťa writes down its remainder when divided by its neighbour in the clockwise direction. Is it possible that all the numbers she wrote down are equal?*

Solution:
As all the numbers are pairwise distinct, one of them has to be the smallest one; let's name it x. We know that both neighbours of x have to be larger than x; let's name the clockwise neighbour c and the anticlockwise neighbour a. We can now calculate $x \bmod c = x$, as x is smaller than c, but $a \bmod x \neq x$, as the remainder has to be smaller than x. This means that we have found a pair where the remainders are different, so all the remainders cannot be equal.

Poznámky:
Většina řešení postupovala podobně jako vzorové a lišila se převážně v tom, zda se dívala na nejmenší nebo na největší prvek v cyklu. *(Káťa Danilina)*

Problem 3.

Consider a sequence $a_1, a_2, \ldots, a_{2025}$ where each term is equal to either 2 or 3. Suppose there exists an integer sequence $b_1, b_2, \ldots, b_{2025}$ satisfying

$$a_i b_i + b_{i+2} \equiv 0 \pmod 5$$

for every $i = 1, 2, \ldots, 2025$ (interpret b_{2026} as b_1 and b_{2027} as b_2). Show that all the numbers $b_1, b_2, \ldots, b_{2025}$ are divisible by 5.

Solution:
For a proof by contradiction, we first assume that the negation is true: There exists an index k such that $b_k \not\equiv 0 \pmod 5$. From the given condition $a_i b_i + b_{i+2} \equiv 0 \pmod 5$, we get $b_{i+2} \equiv -a_i b_i \pmod 5$. Since the a cannot miraculously make the side divisible by 5, we have

$$b_k \not\equiv 0 \pmod 5 \implies b_{k+2} \not\equiv 0 \pmod 5,$$

and this holds for all k.

We observe that every nonzero residue modulo 5 is congruent to a power of 2:

$$2^1 \equiv 2, \quad 2^2 \equiv 4, \quad 2^3 \equiv 3, \quad 2^4 \equiv 1 \pmod 5.$$

The residues 1 and 4 correspond to even powers, while 2 and 3 correspond to odd ones. Multiplying by a number $a_i \equiv \pm 2 \pmod 5$ therefore always swaps between these two sets of residues modulo 5 ($\{1, 4\}$ and $\{2, 3\}$), since it corresponds to adding an odd number (1 or 3) to the exponent. Thus, if $b_i \bmod 5 \in \{1, 4\}$, then $b_{i+2} \bmod 5 \in \{2, 3\}$, and vice versa. Now we can note that $b_k \bmod 5$ and $b_{k+4} \bmod 5$ belong to the same set.

Since $\gcd(2, 2025) = 1$, shifting the index by 2 modulo 2025 passes through all indices exactly once before returning to the starting point. But here we encounter a problem: b_1 and b_{2025} are in the same set (because we "swapped" between the sets n number of times where $4 \mid n$), $b_{2027} = b_2$ is in the other. But b_2 and $b_{2026} = b_1$ must also be in the same set – and b_1 (or any subsequent b_i) cannot belong to both sets at the same time.

The negation is false, therefore the original statement must be true – all b's in the sequence are divisible by 5.

Poznámky:
Většina řešitelů se více či méně (častěji více) komplikovaným způsobem dostala ke zdárnému řešení. Hezky okomentované a vysvětlené postupy dostaly $+i$. *(Lucka Zůnová)*

Problem 4.

Find all pairs of integers (a, b) such that $a^{2026} + (a+b)^{2026} + b^{2026}$ is a perfect square.

Solution:

We want to prove there are no solutions apart from the trivial $(0, 0)$.

Let us suppose (a, b) is a non-trivial solution. Let x be an integer such that both a and b are divisible by 2^x, but at least one of them is not divisible by 2^{x+1}. We will always find suitable x, because we assume at least one of a, b is non-zero. Then let $a = 2^x A$ and $b = 2^x B$.

Now, substituting these into our original expression yiels:

$$a^{2026} + (a+b)^{2026} + b^{2026} = 2^{2026x} A^{2026} + 2^{2026x} (A+B)^{2026} + 2^{2026x} B^{2026} = \\ = 2^{2026x} (A^{2026} + (A+B)^{2026} + B^{2026}).$$

The first factor is a perfect square, therefore, for the whole expression to be a perfect square, the second factor (let us call it S) must be a perfect square as well. We know at least one of A, B is odd. We will look at the expression modulo 4, because perfect squares can only be 0, 1 modulo 4. First, let's look at the case where only one of A, B is odd. WLOG $A = 2n+1$ is odd and $B = 2k$ is even.

$$S = (2n+1)^{2026} + (2n+1+2k)^{2026} + (2k)^{2026} \equiv 1 + 1 + 0 \equiv 2 \pmod 4,$$

which is a contradiction.

Now, consider the case when both $A = 2n+1$, $B = 2k+1$ are odd.

$$S = (2n+1)^{2026} + (2n+1+2k+1)^{2026} + (2k+1)^{2026} \equiv 1 + 0 + 1 \equiv 2 \pmod 4,$$

which once again contradicts the properties of quadratic residues modulo 4.

We proved that S cannot be a perfect square, therefore, there are no any other solutions.

Poznámky:

Většina řešitelů se správně podívala na výraz modulo 4, někteří vytkli mocniny dvojky nebo rovnou největšího společného dělitele, jiní využili metody nekonečného sestupu. Sešla se i nějaká originální řešení, některá došla ku zdárnému konci, někdy se bohužel objevila fatální chyba, poté řešitelé dostali bod za výsledek. Na triviální řešení někteří pozapomněli, ale za to jsem body nestrhával.

(Mára Valkovič)

Problem 5.

A school in PraSe district has grades 1 *through* 12 *with* 100 *students in each grade. As such, there are* 1200 *lockers (numbered* 1 *through* 1200*), one for each student. One day, all the students calculated the remainder their locker number gives when divided by their grade number and compared the results. They would call a grade great if all the students in that grade got the same remainder. Is it possible that the lockers were assigned in a way so that all 12 grades were great?*

Solution:

We show that it is indeed possible. We proceed by induction, showing that if there are $100n$ lockers with any arbitrary numbers on them, we can assign them to n grades so that all of the grades are great.

For $n = 1$, this is statement holds trivially. Now, assume it holds for $n-1$ and we want to prove it for n. Start by assigning the grade n. Using the pigeonhole principle, if we have $100n$ lockers and n remainders, at least one of the holes has at least 100 lockers, then we may assign these lockers to the nth grade. We are left with $100(n-1)$ lockers and $n-1$ grades, a case which already has a solution by our induction hypothesis.

Poznámky:

Většina řešení byla správně. *(Vojta „Dláža“ Gaďurek)*

Problem 6.

The numbers $1, 2, \ldots, 2n$ are divided into two piles of size n. For each of the n^2 ordered pairs in the first pile, Patrik takes the sum of the numbers of that pair, divides it by $2n$ and writes down the resulting remainder on a page in his notebook. He then does the same with pairs from the second pile, writing the results on another page. Prove that both pages contain each remainder the same number of times.

Solution:
We will call the first pile A, the second pile B, and S the set of all numbers 1 through $2n$.

Let us consider some number $x \in S$. We can see that the sequence

$$x + 1, x + 2, \ldots, x + 2n - 1, x + 2n$$

contains each remainder modulo $2n$ exactly once, as it is a sequence of $2n$ consecutive integers. This means that for a given x and a remainder r there exists precisely one number $y \in S$ satisfying the congruence

$$x + y \equiv r \pmod{2n}.$$

So, for an ordered pair containing x to have sum r, it has to be either (x, y) or (y, x). We cannot forget the case where $x = y$, then these two pairs merge into one of the form (x, x). If both x and y belong to the same pile, then r is written down on the corresponding page once for each ordered pair. This means that it is written down twice if x and y are distinct and once if $x = y$, either way exactly once for every number. If, however, x and y belong to different piles, nothing is written down. Let's call these pairs *separated*.

We denote by k the number of separated pairs with sum r. Now, as separated pairs are made up of numbers from different piles, they contain exactly k numbers from A and exactly k numbers from B. This leaves $n - k$ numbers in pile A whose pair is also in pile A and $n - k$ numbers in pile B with pair also in B.

The final piece of the puzzle is just to realize that $n - k$ is also the number of times r is written down on each page (once for every number), which is thus equal on both pages. This proof holds for all remainders modulo $2n$ so we are done.

Poznámky:
Valná většina řešení se alespoň principem shodovala s tím vzorovým, některá k tomu přidala elegantní grafickou reprezentaci ve formě součtové tabulky. *(Anička Trnková)*

Problem 7.

Show that the numbers $1, 3^3, 5^5, \ldots, (2n-1)^{2n-1}$ give pairwise distinct remainders when divided by 2^n.

Solution:
For the sake of contradiction, suppose that

$$(2k-1)^{(2k-1)} \equiv (2\ell-1)^{(2\ell-1)} \pmod{2^n}$$

for some $k, \ell \in \{1, 2, \ldots, n\}$ and $k > \ell$. We will denote $a = 2k-1$ and $b = 2\ell-1$. Substituting gives us $a^a - b^b \equiv 0 \pmod{2^n}$, thus[1] $v_2(a^a - b^b) \geq n$. We can rewrite $a^a - b^b$ as $(a^a - b^a) + (b^a - b^b)$ and study the 2-adic valuations of the two parenthesis separately.

Both a and b are odd, thus $2 \mid a - b$. We can use the LTE lemma on $a^a - b^a$, which gives us

$$v_2(a^a - b^a) = v_2(a - b).$$

[1] By $v_2(n)$ we mean the 2-*adic valuation* of n. You can learn more about it and the LTE lemma here: *https://prase.cz/library/pValuaceMD/pValuaceMD.pdf*.

Because $2 \mid b-1$, we can use the LTE lemma on $b^{a-b}-1$:

$$v_2(b^a-b^b) = v_2(b^b(b^{a-b}-1)) = v_2(b^b)+v_2(b^{a-b}-1) = v_2(b-1)+v_2(b+1)+v_2(a-b)-1.$$

We also used the fact that $v_2(b^b)=0$ because b^b is odd. Both $b-1$ and $b+1$ are even, so

$$v_2(b-1)+v_2(b+1)+v_2(a-b)-1 \geq 1+1+v_2(a-b)-1 = v_2(a-b)+1.$$

Now, from $v_2(a^a-b^a) \neq v_2(b^a-b^b)$, it follows that

$$v_2((a^a-b^a)+(b^a-b^b)) = \min\{v_2(a^a-b^a), v_2(b^a-b^b)\} = v_2(a-b).$$

We are ready to conclude the problem. For all positive integers m we have $v_2(m) \leq \log_2(m)$, therefore $v_2(a-b) \leq \log_2(a-b) \leq \log_2(2n-2) < n$. The second to last inequality follows from $a,b \in \{1,3,\ldots,2n-1\}$ and the last one is equivalent to $2n-2<2^n$, which one can prove using trivial induction. Altogether, we get $n \leq v_2(a^a-b^b) = v_2(a-b) < n$, which is a contradiction.

Poznámky:
Většina správných řešení postupovala podobně jako vzorové, tedy používala LTE lemma. Kladný imaginární bod si vysloužila *Tereza Kubínová*, která ve svém řešení dokazovala silnější tvrzení, že čísla $1^1, 3^3, \ldots, (2^n-1)^{(2^n-1)}$ dávají různé zbytky po dělení 2^n, a to velmi elegantní indukcí.

(David Hromádka)

Problem 8.

Let $p>3$ be a prime such that the smallest positive integer s satisfying $p \mid 2^s-1$ is $s=p-1$. Let $k=\frac{p-3}{2}$. We define a sequence $(a_n)_{n=1}^{\infty}$ by

$$a_i = a_{i+k} = 2^i \quad (1 \leq i \leq k), \qquad a_{j+2k} = a_j a_{j+k} \quad (j \geq 1).$$

Show that there exist $2k$ consecutive terms $a_{x+1}, a_{x+2}, \ldots, a_{x+2k}$ which are pairwise distinct modulo p.

Solution:
First, we note that the condition that the smallest s for which $p \mid 2^s-1$ is $p-1$ means that $p \mid 2^a-2^b$ exactly when $p-1 \mid a-b$. This means that, for every residue modulo $p-1$, the numbers 2^r are distinct modulo p. Because the number of nonzero residues modulo p is the same as the number of all residues modulo $p-1$, this mapping is bijective. Thus, we define b_j as the unique number satisfying $a_j = 2^{b_j}$, which is a natural number because all terms of the sequence a_i are powers of two. From the problem statement, we have

$$a_{j+2k} = a_{j+k}a_j = 2^{b_{j+k}}2^{b_j} = 2^{b_{j+2k}}.$$

This gives the recursion $b_{j+2k} = b_{j+k}+b_j$, where $b_i = b_{i+k} = i$ for $1 \leq i \leq k$.

We can easily observe that the sequence $b_i, b_{i+k}, b_{i+2k}, \ldots$ is just a scaled Fibonacci sequence. So, $b_{jk+i} = iF_j$ for $1 \leq i \leq k$. Considering the Fibonacci sequence modulo $p-1$, we can easily prove that it is periodic. First, the next number can be inferred from the terms F_n and F_{n-1}, and since there are only finitely many pairs modulo $p-1$, the pairs must repeat at some point. To show that it has to be periodic from the start, note that we can get the previous number F_{n-1} from F_n and F_{n+1}. Thus, we can extend F_n backwards from any point, and by the same argument, this has to be periodic from some point, hence it has to be periodic from the start. If $F_1 = F_0 = 1$, we can find $F_{-1} = F_1 - F_0 = 0$, similarly $F_{-2} = 1$, $F_{-3} = -1$.

Now, we just consider at the $2k$ consecutive terms of b_i that are of the form $F_j, 2F_j, \ldots, kF_j, F_{j+1}, \ldots, kF_{j+1}$ where F_j has remainder -1 modulo $p-1$ and F_{j+1} has remainder 1. These are found by noting that F_{-2} and F_{-3} satisfy these requirements, and they have to repeat. Thus these remainders are $p-2, p-3, \ldots, \frac{p+1}{2}, 1, \ldots, \frac{p-3}{2}$. These are $2k$ distinct remainders. Furthermore, the next one is F_{j+2}, which is a repetition of the F_{-1}, so there are $2k+1 = p-2$ distinct remainders, and thus, the a_x terms corresponding to them are $2k+1$ pairwise distinct numbers modulo p.

Poznámky:
Všechna správná řešení nalezla Fibonacciho čísla v exponentech a poté uvažovala rozšíření Fibonacciho čísel zpátky, aby získala posloupnost $-1, 1, 0, 1, 1$. Většina řešení také dokázala opakování dvojic ve Fibonacciho číslech modulo p. Některá zapomněla ukázat to, že posloupnost neobsahuje předperiodu. *(Patrik Štencel)*