Inequalities

4^{TH} AUTUMN SERIES — DATE DUE: 8^{TH} JANUARY 2024

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)

Martin the pig lives with other pigs and chickens on Animal Farm, where all of the animals are supposed to be equal. Some of the animals are members of government and some are not. Martin started to suspect that government members have some advantages. He asked all the pigs and found out that each pig in government gets twice as much food as each pig that is not in government. He asked the chickens and got the same result: any chicken in government gets twice as much food as any chicken outside of government.

He went to Matěj the chief pig and complained. However, Matěj told him: "All of the animals are equal – on average, an animal in government gets the same amount of food as an animal outside of government." Martin was astonished and couldn't believe what he heard.

Show that, despite Martin's disbelief, Matěj could be right.

PROBLEM 2. (3 POINTS)

Let a, b, $c \geq 0$ be real numbers satisfying $a^2 + b^2 = c^2$. Show that

$$\frac{2a+b}{c} \leq \sqrt{5}.$$

PROBLEM 3. (3 POINTS)

One day, Michal brought his favourite square to school and Kuba drew a triangle inside it. Show that Kuba's triangle had at most half the area of Michal's square.

PROBLEM 4. (5 POINTS)

Let m, n be positive integers such that $m + n$ is even. Prove that

$$m! \cdot n! \geq \left(\left(\frac{m+n}{2} \right)! \right)^2 .$$

PROBLEM 5. (5 POINTS)

Let $x, y, z > 0$ be real numbers such that $xy^2z^3 = 108$. Find the smallest possible value of $x+y+z$.

PROBLEM 6. (5 POINTS)

For a positive integer k, let $E(k)$ be the number of even divisors of k and $O(k)$ the number of odd divisors of k. For any positive integer n, prove the inequality

$$\left| \sum_{k=1}^{n} (O(k) - E(k)) \right| \leq n.$$

Problem 7. (5 points)

Three sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ of real numbers satisfy the following equalities for every positive integer n:

$$x_1 = 2, \qquad y_1 = 4, \qquad z_1 = 5,$$

$$x_{n+1} = y_n + \frac{1}{z_n}, \qquad y_{n+1} = z_n + \frac{1}{x_n}, \qquad z_{n+1} = x_n + \frac{1}{y_n}.$$

Prove that $\max(x_n, y_n, z_n) > \sqrt{2n+13}$ for all positive integers n.

Problem 8. (5 points)

Let $\varphi_1, \varphi_2, \varphi_3$ denote the interior angles of a given triangle. Show that

$$\sum_{\text{cyc}} \frac{1}{1 + \cos^2 \varphi_i + \cos^2 \varphi_{i+1}} \leq 2.$$