Introduction to Inequalities for the 4th Autumn series

The problems of this series deal with inequalities, and so this text aims to present some basic methods and important inequalities which can often be helpful in solving these kinds of problems.

Correct reasoning in inequality problems

An inequality problem is usually formulated like this: some variables are given along with some constraints on them (e.g. they are all positive, their sum is equal to something etc.) and we want to show that a given inequality in these variables holds for all values satisfying the constraints. As a first thought, one might usually try to cleverly manipulate some of the algebraic expressions in order to obtain an inequality which is true. That is how we *find* the solution. However, when *writing down* the proof, it is much better (and mathematically rigorous) to go in the exact opposite way – start with facts that are obviously true, and from these conclude that the desired inequality holds.

If all the steps are performed carefully and all the manipulations used are equivalences, there is really no difference, but in more complicated problems, it is easy to unwittingly make a step that is not equivalent, and then the reasoning cannot be reversed – think of the inequality $x^3 \geq 0$, which holds precisely for all nonnegative real x, but squaring both sides yields $x^6 \geq 0$, which is true not only for nonnegative x, but for all real x. Thus, deducing that $x^6 \geq 0$ for nonnegative x from the fact that $x^3 \geq 0$ holds for nonnegative x is perfectly correct, but we cannot reverse the whole argument and say that because $x^6 \geq 0$ for all $x \in \mathbb{R}$, the inequality $x^3 \geq 0$ also holds for every real x.

Manipulating inequalities

Let $x, y, v, w \in \mathbb{R}$. Then, the following statements hold:

(1) If $x < y$ and $u < v$, then $x + u < y + v$.
(2) If $v < x$ and $x < y$, then $v < y$ (the *transitivity* property).
(3) If $x > 0$ and $y > 0$, then $xy > 0$.
(4) If $x < y$ and $a > 0$, then $ax < ay$.
(5) If $x < y$ and $a < 0$, then $ax > ay$.

The mentioned properties are formulated with the *strict inequalities* (i.e. x *is less than* y for $x < y$, or equivalently y *is greater than* x), but they also hold for the *nonstrict inequalities* (that is, x *is less than or equal to* y, which is the same as y *is greater than or equal to* x and which means $x \leq y$).

Important inequalities

We will introduce three famous inequalities that are often utilised in math competitions and which are very important for many fields of mathematics. The first inequality is simple, yet it is often used in the study of real analysis:

Theorem. (Triangle inequality) Let $x, y \in \mathbb{R}$. Then $|x+y| \leq |x| + |y|$.

Proof. For $a \in \mathbb{R}$, $|a|$ is always nonnegative. If $a \geq 0$, then $|a| = a \geq 0 \geq -a$. If $a < 0$, then $|a| = -a > 0 > a$. Hence, for all $a \in \mathbb{R}$, $|a| \geq \pm a$. Thanks to this property, we can prove the triangle inequality.

Let $x, y \in \mathbb{R}$. Then $x + y \geq 0$ or $x + y < 0$. If $x + y \geq 0$, then $|x+y| = x + y$, and because $x \leq |x|$ and $y \leq |y|$, we can sum these inequalities together and get $|x+y| = x + y \leq |x| + |y|$. On the other hand, if $x + y < 0$, then $|x+y| = -(x+y) = -x - y$. Similarly to the previous case, from $-x \leq |x|$, $-y \leq |y|$, we obtain $|x+y| = -x - y \leq |x| + |y|$. □

The next inequality can be generalised for much more abstract objects, but we will formulate it for two finite sequences of real numbers.

Theorem. (Cauchy-Schwarz inequality) Let $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n \in \mathbb{R}$. Then

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right).$$

Proof. Let $P(x) = (a_1x - b_1)^2 + (a_2x - b_2)^2 + \cdots + (a_nx - b_n)^2$. That is a quadratic polynomial and it is clearly nonnegative for every real x, because each of the summands is a square of a real number and hence nonnegative. Therefore, P has at most one real root, and so its discriminant must be less than or equal to zero. We may write P as

$$P(x) = \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)x^2 - 2(a_1b_1 + a_2b_2 + \cdots + a_nb_n)x + \left(b_1^2 + b_2^2 + \cdots + b_n^2\right),$$

and subsequently, the inequality on the discriminant gives us

$$4(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 - 4\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \leq 0,$$

which is equivalent to the Cauchy-Schwarz inequality as stated in the theorem. □

Problem. Let $x_1, x_2, \dots x_n \in \mathbb{R}$. Show that $\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \leq \sqrt{\sum_{i=1}^n x_i^2}$.

Solution. From the CS inequality for $a_i = \frac{1}{\sqrt{n}}$, $b_i = x_i$, $i \in \{1, 2, \dots, n\}$, we have

$$\left(\frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}}\right)^2 \leq \left(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}\right)\left(x_1^2 + x_2^2 + \cdots + x_n^2\right).$$

Then, we can take the square root of both sides (both sides are nonnegative) as

$$\left|\frac{x_1}{\sqrt{n}} + \frac{x_2}{\sqrt{n}} + \cdots + \frac{x_n}{\sqrt{n}}\right| \leq \sqrt{1 \cdot (x_1^2 + x_2^2 + \cdots + x_n^2)},$$

from which (and from the property $a \leq |a|$ for any $a \in \mathbb{R}$) it follows that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n x_i \leq \sqrt{\sum_{i=1}^n x_i^2}.$$

Our last inequality is the most common of all the various inequalities between the so-called *means*. The left-hand side is the *arithmetic mean*, while the right-hand side is the *geometric mean* of the given tuple of numbers:

Theorem. (AM-GM inequality) For $n \in \mathbb{N}$ and nonnegative real numbers $a_1, a_2, \dots, a_n$, the following inequality holds:

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1a_2\cdots a_n}.$$

Problem. For real positive numbers $x_1, x_2, \ldots, x_n$, show that

$$\sum_{\text{cyc}} \frac{x_i}{x_{i+1}} \geq n.$$

The sum on the left-hand side is called a *cyclic sum* and it is just a shorthand for the expression

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1}.$$

Here, we cannot use the more common notation

$$\sum_{i=1}^{n-1} \frac{x_i}{x_{i+1}} = \frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n},$$

because that would omit the last term $\frac{x_n}{x_1}$.

Solution. The numbers $x_1, x_2, \ldots, x_n$ are positive, so each of the fractions $\frac{x_1}{x_2}, \frac{x_2}{x_3}, \ldots, \frac{x_n}{x_1}$ is well-defined and positive. Thus, the following case of the AM-GM inequality holds:

$$\frac{\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_n}{x_1}}{n} \geq \sqrt[n]{\frac{x_1}{x_2} \frac{x_2}{x_3} \cdots \frac{x_n}{x_1}}.$$

The left-hand side's numerator is precisely the cyclic sum from the problem. All the fractions under the right-hand side's n-th root cancel out, as each x_i is present once in the numerator and once in the denominator of the resulting product. Thus, the inequality is equivalent to

$$\frac{\sum_{\text{cyc}} \frac{x_i}{x_{i+1}}}{n} \geq \sqrt[n]{1},$$

which is the same as

$$\sum_{\text{cyc}} \frac{x_i}{x_{i+1}} \geq n,$$

because $n > 0$.