Inequalities

4TH AUTUMN SERIES — DATE DUE: 8TH JANUARY 2024

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)

Martin the pig lives with other pigs and chickens on Animal Farm, where all of the animals are supposed to be equal. Some of the animals are members of government and some are not. Martin started to suspect that government members have some advantages. He asked all the pigs and found out that each pig in government gets twice as much food as each pig that is not in government. He asked the chickens and got the same result: any chicken in government gets twice as much food as any chicken outside of government.

He went to Matěj the chief pig and complained. However, Matěj told him: "All of the animals are equal – on average, an animal in government gets the same amount of food as an animal outside of government." Martin was astonished and couldn't believe what he heard.

Show that, despite Martin's disbelief, Matěj could be right.

PROBLEM 2. (3 POINTS)

Let a, b, $c \geq 0$ be real numbers satisfying $a^2 + b^2 = c^2$. Show that

$$\frac{2a+b}{c} \leq \sqrt{5}.$$

PROBLEM 3. (3 POINTS)

One day, Michal brought his favourite square to school and Kuba drew a triangle inside it. Show that Kuba's triangle had at most half the area of Michal's square.

PROBLEM 4. (5 POINTS)

Let m, n be positive integers such that $m + n$ is even. Prove that

$$m! \cdot n! \geq \left(\left(\frac{m+n}{2}\right)!\right)^2.$$

PROBLEM 5. (5 POINTS)

Let $x, y, z > 0$ be real numbers such that $xy^2z^3 = 108$. Find the smallest possible value of $x+y+z$.

PROBLEM 6. (5 POINTS)

For a positive integer k, let $E(k)$ be the number of even divisors of k and $O(k)$ the number of odd divisors of k. For any positive integer n, prove the inequality

$$\left|\sum_{k=1}^{n}(O(k) - E(k))\right| \leq n.$$

PROBLEM 7. (5 POINTS)

Three sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ of real numbers satisfy the following equalities for every positive integer n:

$$x_1 = 2, \qquad y_1 = 4, \qquad z_1 = 5,$$
$$x_{n+1} = y_n + \frac{1}{z_n}, \qquad y_{n+1} = z_n + \frac{1}{x_n}, \qquad z_{n+1} = x_n + \frac{1}{y_n}.$$

Prove that $\max(x_n, y_n, z_n) > \sqrt{2n+13}$ for all positive integers n.

PROBLEM 8. (5 POINTS)

Let $\varphi_1, \varphi_2, \varphi_3$ denote the interior angles of a given triangle. Show that

$$\sum_{\text{cyc}} \frac{1}{1 + \cos^2 \varphi_i + \cos^2 \varphi_{i+1}} \leq 2.$$

Inequalities

4. podzimní series — Model solutions

Problem 1.

Martin the pig lives with other pigs and chickens on Animal Farm, where all of the animals are supposed to be equal. Some of the animals are members of government and some are not. Martin started to suspect that government members have some advantages. He asked all the pigs and found out that each pig in government gets twice as much food as each pig that is not in government. He asked the chickens and got the same result: any chicken in government gets twice as much food as any chicken outside of government.

He went to Matěj the chief pig and complained. However, Matěj told him: „All of the animals are equal – on average, an animal in government gets the same amount of food as an animal outside of government.“ Martin was astonished and couldn't believe what he heard.

Show that, despite Martin's disbelief, Matěj could be right.

Solution:

The trick is that chickens might eat much less than pigs, but be overrepresented in government. An example of such a situation would be one where chickens get 1 unit of food if they are outside of government and 2 units if they are in government, pigs outside of government get 12 units of food but pigs in government get 24 units and there is 1 pig alongside 10 chickens in government while 3 pigs and 8 chickens are outside of government. The average both in and outside of government would then be

$$\frac{1 \cdot 24 + 10 \cdot 2}{10 + 1} = 4 = \frac{3 \cdot 12 + 8 \cdot 1}{3 + 8},$$

so everything Martin heard could be true.

Poznámky:

Většina řešení byla správně, špatná řešení často pramenila z nedobrého přečtení zadaní. Nakonec jsem se rozhodl uznávat značně pochmurná řešení, že zvířátka nedostávají žádné jídlo. Rád bych však podotknul, že vždy je v případě nalezení trivialního řešení lepší se zeptat, zda bylo záměrem.

(Vojta „Dlážа“ Gaďurek)

Problem 2.

Let a, b, $c \geq 0$ be real numbers satisfying $a^2 + b^2 = c^2$. Show that

$$\frac{2a + b}{c} \leq \sqrt{5}.$$

Solution:

We will prove the given inequality through the use of equivalent manipulations. We can first multiply the inequality by the nonnegative number c:

$$\frac{2a+b}{c} \leq \sqrt{5},$$
$$2a+b \leq \sqrt{5}c.$$

Now because a, b, $c \geq 0$, $2a+b$ is also nonnegative. Since both sides are nonnegative, we can square the inequality:

$$\begin{aligned}(2a+b)^2 &\leq 5c^2,\\ 4a^2+4ab+b^2 &\leq 5c^2,\\ 4a^2+4ab+b^2 &\leq 5a^2+5b^2,\\ 0 &\leq a^2-4ab+4b^2,\\ 0 &\leq (a-2b)^2.\end{aligned}$$

The resulting inequality is always true because the square is always nonnegative.

Poznámky:

Většina řešení postupovala obdobně jako to vzorové. Někteří řešitelé místo úpravy na čtverec použili AG nerovnost nebo Cauchy–Schwarze. Při úpravách nerovností nezapomínejte ukázat, že daná úprava (mocnění, přenásobení neznámou) je v konkrétním případě opravdu ekvivalentní úpravou!

(Klárka Grinerová)

Problem 3.

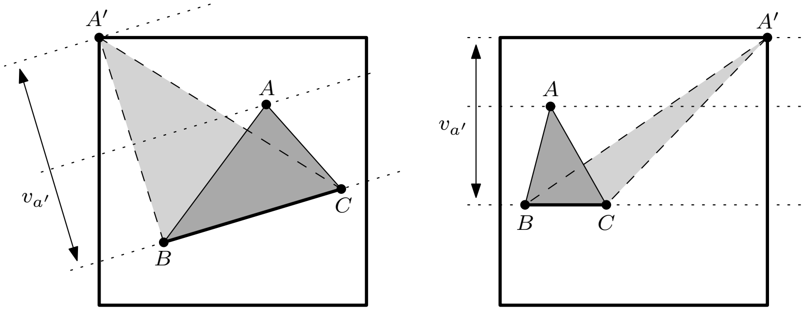
One day, Michal brought his favourite square to school and Kuba drew a triangle inside it. Show that Kuba's triangle had at most half the area of Michal's square.

Solution:

Let us denote Kuba's triangle as ABC. Suppose that the edge BC within the square is fixed. Let us examine which point A' in the square maximizes the area of triangle $A'BC$. The area of $\triangle A'BC$ can be calculated as $\frac{|BC|\cdot v_{a'}}{2}$, where $v_{a'}$ is the height of the triangle with respect to BC.

To maximize the area of $\triangle A'BC$ we thus have to find a point A' in the square such that $v_{a'}$ is maximized. If the line BC is not parallel to one of the sides of the square, then this point is always in one of the square's vertices. If BC is parallel to one side of the square, the point A' can be chosen arbitrarily on the side further from BC. In that case, we can also choose for the point A' to be in the square's vertex.

Therefore, starting with any triangle ABC, we can modify it by moving the point A to A', which we can pick to be in one of the corners of the square in a way that doesn't decrease the area of $\triangle ABC$. When we fix the side AC, we may repeat the same argument and move B to one of the square's corners without decreasing the area, and finally, we do the same with the point C. This way, we end up with a triangle with an area that is at least as large as the original one. Simultaneously however, it has exactly half the area of Michal's square, so the proof is done.



Poznámky:
Správná řešení vesměs postupovala podobným způsobem jako vzorové řešení. Často ale používala krok navíc, kde se nejprve odargumentovalo, že vrcholy trojúhelníku musí pro maximální obsah ležet na stranách čtverce, a až poté se dokázalo, že je můžeme přesunout do vrcholů. Někteří řešitelé rozebírali různé konfigurace, v jakých se trojúhelník může nacházet. Ovšem na nějaké z nich občas zapomněli. *(Lenka Kopfová)*

Problem 4.

Let m, n be positive integers such that $m+n$ is even. Prove that

$$m!\cdot n! \geq \left(\left(\frac{m+n}{2}\right)!\right)^2.$$

Solution:
First, we can see that the inequality holds when $m=n$, since

$$m!\cdot m! = (m!)^2 = \left(\left(\frac{2m}{2}\right)!\right)^2.$$

Now we can assume WLOG that $m>n$. Let $k=\frac{m+n}{2}$, which we know is an integer. Then we can write $m=k+\ell$ and $n=k-\ell$ for some $\ell\in\mathbb{N}$. With this, we restate the inequality we want to prove as

$$(k+\ell)!\cdot(k-\ell)! \geq (k!)^2,$$

which is equivalent to

$$k!\cdot(k+1)(k+2)\cdots(k+\ell)\cdot\frac{k!}{(k-\ell+1)(k-\ell+2)\cdots k} \geq (k!)^2,$$
$$(k!)^2\cdot\frac{(k+1)(k+2)\cdots(k+\ell)}{(k-\ell+1)(k-\ell+2)\cdots k} \geq (k!)^2.$$

Dividing both sides by $(k!)^2$, we get an equivalent inequality

$$\frac{(k+1)(k+2)\cdots(k+\ell)}{(k-\ell+1)(k-\ell+2)\cdots k} \geq 1.$$

Since the numerator and denominator on the left-hand side both contain ℓ terms, we may rewrite the fraction as

$$\frac{k+1}{k-\ell+1}\cdot\frac{k+2}{k-\ell+2}\cdots\frac{k+\ell}{k} \geq 1.$$

Now, if we look at the individual ratios, we see that each is greater than 1, since $k>k-\ell$. Therefore their product is also greater than 1, which is what we wanted to prove.

Poznámky:
Většina řešení se vydala cestou vzorového a skoro vždycky došla k cíli. Dorazilo i pár řešení, co postupovala indukcí. *(Káťa Danilina)*

Problem 5.

Let $x, y, z > 0$ be real numbers such that $xy^2z^3 = 108$. Find the smallest possible value of $x+y+z$.

Solution:
The numbers x, y, and z are all real and positive. Therefore, we can use the AM-GM inequality. The product's value $xy^2z^3 = 108$ is the only number we have, so we want to use it in our estimate. Therefore, we want to apply AM-GM on a collection of numbers with one x, two y's, and three z's for the expression xy^2z^3 to appear in the geometric mean. The easiest way to do this and still estimate the sum of the original three variables is to express $y = \frac{y}{2} + \frac{y}{2}$, and $z = \frac{z}{3} + \frac{z}{3} + \frac{z}{3}$. This yields

$$x + y + z = x + \frac{y}{2} + \frac{y}{2} + \frac{z}{3} + \frac{z}{3} + \frac{z}{3} \geq 6 \cdot \sqrt[6]{x \cdot \frac{y}{2} \cdot \frac{y}{2} \cdot \frac{z}{3} \cdot \frac{z}{3} \cdot \frac{z}{3}} = 6 \cdot \sqrt[6]{\frac{1}{108}xy^2z^3} = 6.$$

Thus, the value of $x + y + z$ is at least 6. All that's left to prove is that $x + y + z = 6$. Equality occurs in the AM-GM inequality if and only if all the arithmetic/geometric mean inputs are equal. In our case, this happens when

$$x + y + z = 6 \qquad \text{and} \qquad x = \frac{y}{2} = \frac{z}{3},$$

which has the solution

$$x = 1, \quad y = 2, \quad z = 3.$$

This triplet also satisfies

$$xy^2z^3 = 1 \cdot 2^2 \cdot 3^3 = 108,$$

so all conditions are met, and the smallest possible value is 6.

Poznámky:
Většina řešitelů se více či méně ubírala podobnou cestou jako vzorové řešení. Někteří AGčko použili na zbytečně složité výrazy, ale i tak se úspěšně dobrali k cíli. Pár jedinců se zamotalo, někdo jenom máchal rukama a takovým jsem dávala jeden bod za správné, ale nepodložené řešení. Z čeho jsem však smutná, je, že kolem desítky řešitelů se ubíralo cestou ne úplně pěkných, analytických řešení, čímž si vysloužili $-i$. Nemálo z nich pak ztroskotalo na nějakém pořádnějším argumentování, a proto prosím, nepouštějte se do analytických řešení, když to neumíte! Naše příklady mají vždy i nějaké jiné, typicky hezčí, řešení. :) *(Adéla Karolína „Áďa“ Žáčková)*

Problem 6.

For a positive integer k, let $E(k)$ be the number of even divisors of k and $O(k)$ the number of odd divisors of k. For any positive integer n, prove the inequality

$$\left| \sum_{k=1}^{n} (O(k) - E(k)) \right| \leq n.$$

Solution:
The first step is to examine the sum $\sum_{k=1}^{n} O(k)$. It counts the number of odd divisors of each natural number k from 1 to n. Note that this is the same as if, for each odd number d from 1 to n, we counted how many numbers from 1 to n are divisible by d. For a given d, this amount is equal to $\lfloor \frac{n}{d} \rfloor$, which means we can rewrite the sum as

$$\sum_{k=1}^{n} O(k) = \left\lfloor \frac{n}{1} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor + \cdots,$$

with the right-hand side ending with either $\lfloor \frac{n}{n-1} \rfloor$ or $\lfloor \frac{n}{n} \rfloor$, depending on the parity of n.

Applying the same reasoning for the sum $\sum_{k=1}^{n} E(k)$ and getting rid of the absolute value, we can rewrite the initial inequality as

$$-n \leq \left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{4} \right\rfloor + \cdots \leq n. \tag{1}$$

The key to proving the inequalities above is to realize that

$$\left\lfloor \frac{n}{d} \right\rfloor \geq \left\lfloor \frac{n}{d+1} \right\rfloor.$$

With this tool in hand, let us prove both inequalities in (1).

If n is even, we see that

$$\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \cdots + \left\lfloor \frac{n}{n-1} \right\rfloor - \left\lfloor \frac{n}{n} \right\rfloor \geq \left(\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right) + \cdots + \left(\left\lfloor \frac{n}{n} \right\rfloor - \left\lfloor \frac{n}{n} \right\rfloor \right) = 0,$$

since all the positive and negative terms have paired up. One can also see that

$$\left\lfloor \frac{n}{1} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor + \cdots - \left\lfloor \frac{n}{n} \right\rfloor \leq \left\lfloor \frac{n}{1} \right\rfloor + \left(- \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor \right) + \cdots + \left(- \left\lfloor \frac{n}{n-1} \right\rfloor + \left\lfloor \frac{n}{n-1} \right\rfloor \right) - \left\lfloor \frac{n}{n} \right\rfloor = n - 1.$$

For odd n, it works in much the same way. The only difference is that in the first inequality, the last term $+\lfloor \frac{n}{n} \rfloor$ is left unpaired, meaning the whole expression is ≥ 1. And in the second inequality all the negative terms have a positive term to pair up with, so the whole expression is $\leq n$. We have, therefore, finished the proof.

Poznámky:
Skoro všechna došlá řešení postupovala v duchu řešení vzorového (sumu si vyjádřila druhým způsobem a pomocí klíčového poznatku dokázala kýžené nerovnosti) a vysloužila si plný počet bodů. Velká pochvala putuje *Tomáši Pazourkovi*, který dokázal napsat kompletní řešení jen v několika málo řádcích tím, že poznal konvergentní řadu, a vysloužil si tak bonusové $+i$.

(Josef „José“ Soural)

Problem 7.

Three sequences $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ of real numbers satisfy the following equalities for every positive integer n:

$$x_1 = 2, \qquad y_1 = 4, \qquad z_1 = 5,$$

$$x_{n+1} = y_n + \frac{1}{z_n}, \qquad y_{n+1} = z_n + \frac{1}{x_n}, \qquad z_{n+1} = x_n + \frac{1}{y_n}.$$

Prove that $\max(x_n, y_n, z_n) > \sqrt{2n+13}$ for all positive integers n.

Solution:

We will show the desired conclusion by induction with respect to n. The base case $n = 1$ is clear since $z_1 = 5 > \sqrt{15}$.

Now, let us assume that for some n we have $\max(x_n, y_n, z_n) > \sqrt{2n + 13}$. We will derive a lower bound on $\max(x_{n+1}, y_{n+1}, z_{n+1})$. This expression is, by definition, at least as large as any of the numbers x_{n+1}, y_{n+1}, and z_{n+1}. Using the recurrence relations defining these sequences, we will pick the sequence where the non-reciprocated term (i.e. y_n in $y_n + \frac{1}{z_n}$ etc.) is equal to $\max(x_n, y_n, z_n)$. For some $a \in \{x_n, y_n, z_n\}$, we then have

$$\max(x_{n+1}, y_{n+1}, z_{n+1}) \geq \max(x_n, y_n, z_n) + \frac{1}{a} \geq \max(x_n, y_n, z_n) + \frac{1}{\max(x_n, y_n, z_n)},$$

since $a \leq \max(x_n, y_n, z_n)$ and all the terms of the sequence are positive. Therefore

$$\max(x_{n+1}, y_{n+1}, z_{n+1})^2 \geq \left(\max(x_n, y_n, z_n) + \frac{1}{\max(x_n, y_n, z_n)}\right)^2 > \max(x_n, y_n, z_n)^2 + 2,$$

by expanding. Using the induction hypothesis, we get

$$\max(x_{n+1}, y_{n+1}, z_{n+1})^2 \geq \max(x_n, y_n, z_n)^2 + 2 > 2n + 13 + 2 = 2(n + 1) + 13.$$

Finally, since all the terms are positive, we have $\max(x_{n+1}, y_{n+1}, z_{n+1}) > \sqrt{2(n + 1) + 13}$, as we wanted.

Alternative solution:

The following solution is based on studying the sums of squares of the n-th terms of the respective sequences. We will then show that the following inequality holds:

$$x_n^2 + y_n^2 + z_n^2 \geq 6n + 39.$$

To this end, proceed by induction on n. The base case $n = 1$ is clear since $x_1^2 + y_1^2 + z_1^2 = 45 \geq 45$. Now assume that for some $n \in \mathbb{N}$ we have $x_n^2 + y_n^2 + z_n^2 \geq 6n + 39$. For the induction step, we simply square the recurrence relations defining our sequence and use the induction hypothesis:

$$\begin{aligned}
x_{n+1}^2 + y_{n+1}^2 + z_{n+1}^2 &= \left(y_n + \frac{1}{z_n}\right)^2 + \left(z_n + \frac{1}{x_n}\right)^2 + \left(x_n + \frac{1}{y_n}\right)^2 = \\
&= x_n^2 + y_n^2 + z_n^2 + 2\left(\frac{y_n}{z_n} + \frac{z_n}{x_n} + \frac{x_n}{y_n}\right) + \frac{1}{x_n^2} + \frac{1}{y_n^2} + \frac{1}{z_n^2} \geq \\
&\geq 6n + 39 + 2\left(\frac{y_n}{z_n} + \frac{z_n}{x_n} + \frac{x_n}{y_n}\right) + \frac{1}{x_n^2} + \frac{1}{y_n^2} + \frac{1}{z_n^2} > \\
&> 6n + 39 + 2\left(\frac{y_n}{z_n} + \frac{z_n}{x_n} + \frac{x_n}{y_n}\right) \geq 6n + 39 + 6 = 6n + 45,
\end{aligned}$$

where the last line follows AM-GM on the three fractions $\frac{y_n}{z_n}$, $\frac{z_n}{x_n}$, and $\frac{x_n}{y_n}$. Therefore, for $n > 1$, we have the sharp inequality

$$x_{n+1}^2 + y_{n+1}^2 + z_{n+1}^2 > 6n + 45 = 6(n + 1) + 39,$$

as claimed. Therefore, for $n > 1$, we can finally conclude with

$$\max(x_n, y_n, z_n)^2 \geq \frac{x_n^2 + y_n^2 + z_n^2}{3} > \frac{6n + 39}{3} = 2n + 13,$$

so $\max(x_n, y_n, z_n) > \sqrt{2n + 13}$, since the sequences contain only positive terms. Thus we are done.

Poznámky:

Většina řešení postupovala vesměs podle jednoho z uvedených řešení a mimo malé chybky byla skoro všechna správně. *(Zdeněk Pezlar)*

Problem 8.

Let $\varphi_1, \varphi_2, \varphi_3$ denote the interior angles of a given triangle. Show that

$$\sum_{\text{cyc}} \frac{1}{1+\cos^2\varphi_i+\cos^2\varphi_{i+1}} \leq 2.$$

Solution:

Because $\varphi_1, \varphi_2, \varphi_3$ are the interior angles of a triangle, we have (cyclically for $i \in \{1,2,3\}$)

$$\varphi_i + \varphi_{i+1} = \pi - \varphi_{i+2}.$$

For any real x, it holds that $\sin(\pi - x) = \sin x$, so

$$\sin\varphi_{i+2} = \sin(\varphi_i+\varphi_{i+1}) = \sin\varphi_i\cos\varphi_{i+1} + \cos\varphi_i\sin\varphi_{i+1}$$

using the sine addition formula. We can square both sides of the equation and apply the Cauchy-Schwarz inequality to the right-hand side:

$$\sin^2\varphi_{i+2} = (\sin\varphi_i\cos\varphi_{i+1} + \cos\varphi_i\sin\varphi_{i+1})^2 \leq (\sin^2\varphi_i + \sin^2\varphi_{i+1})(\cos^2\varphi_{i+1} + \cos^2\varphi_i).$$

The angles of the triangle are all strictly smaller than π and greater than 0 – otherwise, the triangle would be degenerate. It follows that $\sin\varphi_1, \sin\varphi_2$ and $\sin\varphi_3$ are all positive. In particular, the value $\sin^2\varphi_i + \sin^2\varphi_{i+1} > 0$. Therefore, we can divide both sides of the previous inequality without changing the inequality sign:

$$\begin{aligned}
\frac{\sin^2\varphi_{i+2}}{\sin^2\varphi_i+\sin^2\varphi_{i+1}} &\leq \cos^2\varphi_i+\cos^2\varphi_{i+1}, \\
1+\frac{\sin^2\varphi_{i+2}}{\sin^2\varphi_i+\sin^2\varphi_{i+1}} &\leq 1+\cos^2\varphi_i+\cos^2\varphi_{i+1}, \\
\frac{\sin^2\varphi_i+\sin^2\varphi_{i+1}+\sin^2\varphi_{i+2}}{\sin^2\varphi_i+\sin^2\varphi_{i+1}} &\leq 1+\cos^2\varphi_i+\cos^2\varphi_{i+1}.
\end{aligned}$$

Both sides of the resulting inequality are larger than zero, so we can divide the inequality by both of them to obtain

$$\frac{1}{1+\cos^2\varphi_i+\cos^2\varphi_{i+1}} \leq \frac{\sin^2\varphi_i+\sin^2\varphi_{i+1}}{\sin^2\varphi_i+\sin^2\varphi_{i+1}+\sin^2\varphi_{i+2}}.$$

Finally, using the cyclic sum, we get

$$\begin{aligned}
\sum_{\text{cyc}} \frac{1}{1+\cos^2\varphi_i+\cos^2\varphi_{i+1}} &\leq \sum_{\text{cyc}} \frac{\sin^2\varphi_i+\sin^2\varphi_{i+1}}{\sin^2\varphi_i+\sin^2\varphi_{i+1}+\sin^2\varphi_{i+2}}, \\
\sum_{\text{cyc}} \frac{1}{1+\cos^2\varphi_i+\cos^2\varphi_{i+1}} &\leq \frac{2\sin^2\varphi_i+2\sin^2\varphi_{i+1}+2\sin^2\varphi_{i+2}}{\sin^2\varphi_i+\sin^2\varphi_{i+1}+\sin^2\varphi_{i+2}} = 2,
\end{aligned}$$

Poznámky:

Úloha byla těžší triková, a řešení proto nepřišlo mnoho. Správná řešení postupovala velmi podobným postupem, kdy se čtverce kosinů vyjádřily z kosinové věty pomocí stran trojúhelníku a na ně se aplikovala CS nerovnost. Řešení *Patrika Štencela* se dokonce úplně bez CS nerovnosti obešlo a postupovalo jen pomocí goniometrických vztahů a náhledu, že cyklická suma nabývá maxima právě pro trojice úhlů, z nichž dva jsou shodné. *(Matěj Gajdoš)*