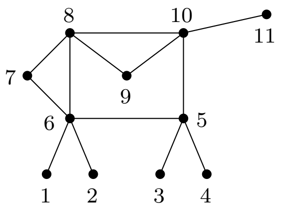
Introduction to the 4th autumn series

This year's 4th autumn series revolves around communication, which can take many shapes and involve quite a diverse range of mathematical concepts. The notion of a *graph*, which we'll overview in this introductory text, is one such concept. Our aim is to provide you with some terminology to talk about graphs and with basic results and tricks.

What is a graph?

Graphs come in many different flavours, so don't be surprised if you encounter slightly altered setups elsewhere. Let us start with just a basic *graph*[1]. It consists of some *vertices*[2] and some *edges* that connect them – each edge connects two of the vertices (we call them the *endpoints* of this edge). Formally, we can say that there is some finite set of vertices V and that the set of edges E contains some of the two-element subsets of V.

To visualize graphs, we usually draw vertices as dots and edges as lines connecting them, like this:



$$V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\},$$
$$E = \Big\{\{1,6\}, \{2,6\}, \{3,5\}, \{4,5\}, \{5,6\}, \{6,7\}, \{7,8\}, \{6,8\}, \{8,9\}, \{8,10\}, \{5,10\}, \{9,10\}, \{10,11\}\Big\}$$

As we've said, you might encounter many variations of the rules, but for us, graphs won't be allowed to have *repeated edges* (multiple edges with the same endpoints) or *loops* (edges that connect a vertex with itself).

The usefulness of graphs comes from their ability to represent many situations. Usually, the vertices represent some objects (people, towns, airports, ...), whilst edges provide some kind of connection or relation between them (friendships, roads, airlines, ...).

Degree, paths, cycles and cliques

It is often useful to consider how many edges have an endpoint in some given vertex v. This quantity is called the *degree* of v and we denote it $\deg v$.

Proposition. (handshaking lemma) Consider a graph with vertices $V = \{v_1, \ldots, v_n\}$ and a set E of edges between them. Then $\deg v_1 + \deg v_2 + \cdots + \deg v_n = 2 \cdot |E|$.

Proof. Imagine each edge pays one coin to each of its endpoints. It is then clear that each edge spent 2 coins, so the total amount paid will be $2 \cdot |E|$. On the other hand, each vertex v received one coin for each of its edges, meaning it received $\deg v$ coins. Adding this up over all vertices, the total amount paid must also equal the sum of degrees of all vertices. This proves the proposition. □

As a consequence, this implies that the number of vertices with odd degrees has to be even in any graph.

[1] Sometimes called an *undirected graph* to distinguish it from a *directed graph*, which we'll see later.

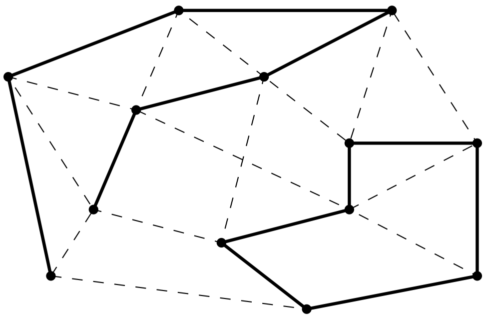
[2] It's one *vertex*, but many *vertices*.

If we view a graph as a map of towns and roads between them, we might ask ourselves what kind of a road trip we can plan. This leads to concepts like paths and cycles.

A *path* between vertices a and b is what you might expect: a sequence of edges that starts at a, ends at b and everywhere in the middle, the ending point of one edge is the starting point of the next. Formally, we may say it is a sequence of edges

$$\{v_0, v_1\},\ \{v_1, v_2\},\ \{v_2, v_3\},\ \ldots,\ \{v_{k-1}, v_k\}$$

such that $v_0 = a$, $v_k = b$. We don't allow a vertex to be visited more than once.



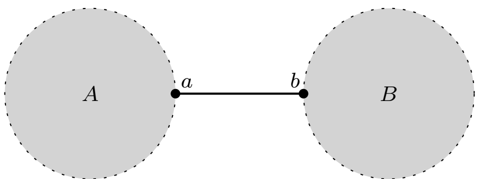
A *cycle* is similar to a path, except we require our sequence of edges to come back to the starting vertex. Think of it as a round trip where you tour some towns and then come back home. Again, no town except your hometown is allowed to be visited repeatedly.

A *clique* is a group of vertices that are all connected with each other by edges. In a graph where vertices are people and edges are friendships between them, a clique is a group where everyone is friends with one another.

We say a graph is *connected* if there is a path between any two vertices. A *complete graph* is one where there is an edge between any two vertices, i.e. all vertices form one big clique.

Problem. Suppose that we have a connected graph where all vertices have even degrees. Prove that if we delete any single edge, the graph will stay connected.

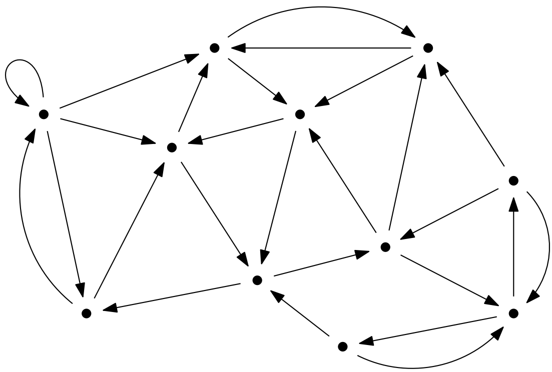
Solution. For the sake of contradiction, suppose that when we delete the edge connecting some vertices a and b, the graph becomes disconnected. This means we are left with two parts of the graph: A, which contains a, and B, which contains b.



Let us look at A as a separate graph. Notice that all vertices except for a have an even degree, because they had even degrees in the original graph and didn't lose any edges. On the other hand, a lost precisely one edge, so its degree must now be odd. This would mean that the sum of degrees in A is now odd, which contradicts the handshaking lemma. This is the desired contradiction, so the graph stays connected even after we delete the edge between a and b.

Directed graphs

Finally, let us talk briefly about directed graphs. If the undirected graphs represented symmetric relations (friendship, connecting road, …), directed graphs represent asymmetric relations (parent–child, boss–employee, one-way road, …). Formally speaking, we have once again a set V of vertices and a set E of edges, but this time, edges are ordered pairs of vertices. This time, it is allowed for two vertices to be connected in both directions or for an edge to connect a vertex to itself. We usually draw these oriented edges as arrows and we distinguish their starting point from their endpoint.



In a directed graph, we can still define the degree of a vertex v, but we need to distinguish the *indegree* $\deg_{\text{in}} v$ (how many arrows point into v) from the *outdegree* $\deg_{\text{out}} v$ (how many arrows point out from v). With these notions, we have a lemma akin to the handshaking lemma:

Proposition. Any directed graph with vertices $V = \{v_1, \dots, v_n\}$ and edges E satisfies

$$\deg_{\text{in}}(v_1) + \deg_{\text{in}}(v_2) + \dots + \deg_{\text{in}}(v_n) = \deg_{\text{out}}(v_1) + \deg_{\text{out}}(v_2) + \dots + \deg_{\text{out}}(v_n) = |E|.$$

Proof. The proof is very similar to that of the handshaking lemma. This time, imagine each oriented edge pays its starting point a red coin and its endpoint a blue coin. Then the total amount of red coins and the total amount of blue coins will be the same – both are equal to the number of edges. But every vertex v receives $\deg_{\text{out}}(v)$ red coins and $\deg_{\text{in}}(v)$ blue coins. So the right and left hand sides of the identity we wish to prove equal the total number of blue and red coins respectively, and they are both equal to $|E|$. □

Hopefully, this gives you enough of a head start on the 4$^{\text{th}}$ autumn series. If you want to know more about graphs, we have a serial (written in Czech) from a few years ago that can give you a more thorough introduction to all things graph-related. You can find it at *https://prase.cz/archive/34/serial.pdf*. In any case, good luck with the problems!