Communication

4th autumn series Date due: 9th January 2023

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

Problem 1. (3 points)

A group of 26 pupils went to Matfyz. Every pupil is either always telling the truth or always lying. On their way there they went in pairs and everyone said "My partner is a liar." Could they have rearranged themselves in pairs, so that everyone said "My partner is always telling the truth"?

Problem 2. (3 points)

Petr, Denisa and Markéta are playing a game. First, Denisa tells Petr seven pairwise distinct positive integers whose sum is 100. Petr then tells Markéta the fourth highest of them and Markéta has to guess all seven numbers. The girls win if Markéta guesses all numbers correctly, but they cannot discuss their strategy beforehand. Which numbers should Denisa choose in order to win?

Problem 3. (3 points)

There are n piglets and each of them created one unique meme. One piglet can send a message to another containing all memes he currently knows. How many messages must be sent so that every piglet knows every meme?

Problem 4. (5 points)

Twenty teams competed in a tournament. On the first day, each team played (exactly) one game against one of the other teams. On the second day, every team played one game again, but this time against a different opponent. Prove that on the third day of the tournament, we can still choose ten teams such that no two of them played together before.

Problem 5. (5 points)

Distances between pairs of cities in PraSestan are unique.[1] Out of each city, an airplane departs to its nearest neighbor. Find the maximum number of airplanes which can end up in the same city.

Problem 6. (5 points)

An odd number of teams entered a two-day tournament. On the first day, every team faced every other team. The next day, they faced every other team again. Results of the tournament satisfy the following condition: each team lost as many times as it won and there were no draws. Prove that some half of the matches can be ignored so that the condition will still hold.

Problem 7. (5 points)

On Christmas Eve, n piglets exchanged some Christmas GIFs. We know that for each pair of piglets, at least one of them sent the other a Christmas GIF. We also know that every piglet received a Christmas GIF from exactly one quarter of the recipients of his Christmas GIFs. Determine all possible values of n.

[1] PraSestan can be represented as a plane where cities are points.

Problem 8. (5 points)

In the kingdom of Esarpia, there live $n \geq 3$ peasants. Some pairs of these peasants are friends[2]. We say that four peasants a, b, c and d form a *friendly square*, if the four pairs $\{a, b\}$, $\{b, c\}$, $\{c, d\}$ and $\{d, a\}$ are friends, but the pairs $\{a, c\}$ and $\{b, d\}$ are not.

King Esarp enacted a decree that a several day long festival is to be held to celebrate the birthday of his most trusted Archmage. The decree stipulates the following rules:

- On each day of the festival, a party must be held. Once no more parties can be held, the festival ends.
- A group of peasants may hold a party, only if they are all friends with each other and no peasant that does not participate is friends with all the participants.
- No two parties can have precisely the same set of participants.

Prove that if there are no friendly squares in Esarpia, then the festival can last at most $\frac{n(n-1)}{2}$ days.

[2]Friendship is symmetric.

Communication

4^{TH} AUTUMN SERIES MODEL SOLUTIONS

Problem 1.

A group of 26 pupils went to Matfyz. Every pupil is either always telling the truth or always lying. On their way there they went in pairs and everyone said „My partner is a liar.“ Could they have rearranged themselves in pairs, so that everyone said „My partner is always telling the truth“?

(Adéla Karolína Žáčková)

SOLUTION:
We shall prove that the pupils couldn't have rearranged themselves in pairs so that everyone said "My partner is always telling the truth".

There are three possible kinds of pairs. In the first kind of pair – truth-teller and truth-teller – they both always tell the truth, so they both say "My partner is always telling the truth". In the second kind – truth-teller and liar – they both end up saying "My partner is a liar", which is a lie for the liar and true for the truthful one. In the third kind – liar and liar – they both lie, so they say "My partner is always telling the truth".

Since everyone said "My partner is a liar", we can conclude that every pair consists of one liar and one truthful pupil. There are 26 pupils, which means that 13 of them are liars and 13 of them are truth-tellers. If they wanted to rearrange themselves so that everyone says "My partner is always telling the truth", they would need to be in pairs where either both are truthful or both are liars. But that is clearly not possible, because there is an odd number of truth-tellers and an odd number of liars.

POZNÁMKY:
Téměř všechna řešení byla správná a postupovala stejně jako vzorové řešení.

(Klárka Grinerová)

Problem 2.

Petr, Denisa and Markéta are playing a game. First, Denisa tells Petr seven pairwise distinct positive integers whose sum is 100. Petr then tells Markéta the fourth highest of them and Markéta has to guess all seven numbers. The girls win if Markéta guesses all numbers correctly, but they cannot discuss their strategy beforehand. Which numbers should Denisa choose in order to win?

(Magdaléna Mišinová)

SOLUTION:
The numbers Denisa should choose are 1, 2, 3, 22, 23, 24, 25 (whose sum is 100). This way, if Petr gives Markéta the fourth highest number, i.e. 22, the rest are uniquely determined. Why? The sum of the three smallest natural numbers (1, 2, 3) is 6 and the sum of the three smallest numbers greater than 22 (23, 24, 25) is 72. Together, they sum to $6 + 22 + 72 = 100$. Therefore, if we are given the number 22, the rest of the numbers must be the smallest possible ones, since otherwise the sum would become larger than 100.

Poznámky:
Naprostá většina řešení byla správná a hezky odůvodněná. Těch pár řešení, která správná nebyla, alespoň měla myšlenku „udělejme malé/velké členy co nejmenší/největší“, za což jsem nějaké body dávala. Ve dvou případech bohužel bylo opomenuto, že čísla mají být rozdílná, ale i v těchto řešeních byly nějaké správné myšlenky, za které jsem body dala. *(Anna Marie Minarovičová)*

Problem 3.

There are n piglets and each of them created one unique meme. One piglet can send a message to another containing all memes he currently knows. How many messages must be sent so that every piglet knows every meme? *(Klátra)*

Solution:
There has to be a time when exactly one piglet knows every meme. That is because a message is sent between one sender and one receiver. After such a piglet exists, every other piglet has to get at least one more message to get to know every meme. That means at least $n-1$ messages must be sent after some piglet knows all the memes.

To get one piglet to know every meme, every other piglet had to send at least one message containing its meme. Otherwise, his meme would not get to the all-knowing piglet. That means at least $n-1$ different messages sent.

In summary, at least $n-1$ messages were sent to get one piglet to know every meme and at least $n-1$ messages afterward to get every piglet to know every meme from the piglet who knows all the memes. This means that overall, at least $2(n-1)$ messages must have been sent.

To achieve that every piglet knows every meme in $2n-2$ messages, piglets can follow a simple algorithm:

Every piglet except piglet P sends his meme to piglet P. Then, piglet P sends messages containing all memes to every other piglet. With this algorithm, $2n-2$ messages were sent. Furthermore, we have shown above that less messages would not be sufficient, so this construction is optimal and we are done.

Poznámky:
Hodně řešitelů našlo správné řešení i algoritmus, jak se k němu dojde. Ale ještě k tomu je nutné, abyste dokázali, že to opravdu je nejmenší možný počet. *(Petr Hladík)*

Problem 4.

Twenty teams competed in a tournament. On the first day, each team played (exactly) one game against one of the other teams. On the second day, every team played one game again, but this time against a different opponent. Prove that on the third day of the tournament, we can still choose ten teams such that no two of them played together before. *(Magdaléna Mišinová)*

Solution:
We start by constructing a graph where vertices are teams and edges are the matches played. It is easy to see that every vertex has exactly two edges incident to it. This implies that every connected component is a cycle. We can prove this by choosing any vertex and then travelling through the graph, starting at the chosen vertex and not allowing ourselves to use any edge twice. Because every vertex has two edges coming out of it, our journey must end at the starting vertex.

Next, we will prove that every cycle has an even length. In order to see this, we colour an edge white if the corresponding match was played on the first day and black if it was played on the second day. So there will be one black and one white edge adjacent to every vertex. However, then there must be an even number of edges in any given cycle, since the edge colours clearly alternate.

As every cycle is even, we can choose every other vertex from each cycle. Thus we get precisely half the vertices from any cycle, so ten teams in total. No edges can exist between the chosen vertices, so this choice satisfies the conditions from the problem statement.

Poznámky:
Všechna správná řešení postupovala obdobně. Někteří řešitelé bohužel neřešili problém obecně a odevzdali pouze konstrukci pro nějaký případ. Taková řešení získala jen velmi malý počet bodů.

(Vojta „Dláža“ Gaďurek)

Problem 5.

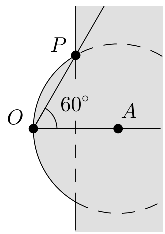
Distances between pairs of cities in PraSestan are unique.[1] *Out of each city, an airplane departs to its nearest neighbor. Find the maximum number of airplanes which can end up in the same city.*

(Marian Poljak)

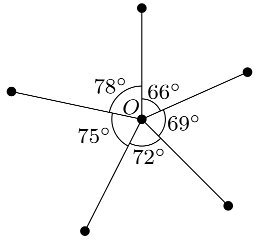
Solution:
Let O be a city in PraSestan. Our objective is to place as many cities as possible around O so that the airplanes departing from those cities end up in O, and so that the distance between any pair of cities is unique. Let us place a point A which does not collide with O. The airplane from A must land in O. Now we would like to place B, so that an airplane departing from B lands in O without changing the course of A's airplane. This implies two pieces of information:

(1) B cannot lie inside the circle k centred at A with radius $|AO|$, otherwise the airplane from A would end up in B.
(2) B cannot lie in the half-plane defined by the bisector of the line segment $\overline{AO}$ and A. If that was the case, the airplane from B would land in A.

Hence we have a region where B cannot be placed. This region must include the boundary of k and the bisector of $\overline{AO}$, since otherwise we would contradict our assumption that all distances are unique. The bisector of $\overline{AO}$ intersects k in two points. We will call one of them P. Since the bisector of $\overline{AO}$ represents the set of all points Z for which $|AZ| = |OZ|$, we have $|AP| = |AO| = |OP|$. Therefore $\triangle AOP$ is an equilateral triangle and $|\angle AOP| = 60°$. Now, if we place B so that $|\angle AOB| \leq 60°$, B will surely lie in the forbidden region. This logic holds for any pair of distinct points X and Y around O, so we must have $|\angle XOY| > 60°$. It follows that we can place at most 5 cities around O as 6 or more will not fit (since $6 \cdot |\angle XOY| > 360°$).



We have shown that the maximum number of airplanes which can land in the same city cannot be more than 5. The following arrangement of exactly 5 points around O works, so 5 is the number we are looking for.



[1] PraSestan can be represented as a plane where cities are points.

Poznámky:
Téměř všechna kompletně správná řešení se řídila podobnou strukturou. Nicméně, mnohdy se stávalo, že řešitel dobře dokázal proč 6 a více bodů nelze položit kolem jednoho města podle našich požadavků, ale pak nikde nebyla zahrnutá konstrukce (náznak bohatě stačil) pěti měst. Za to se strhával bod, jelikož takový důkaz je neúplný. *(Diana Bergerová)*

Problem 6.

An odd number of teams entered a two-day tournament. On the first day, every team faced every other team. The next day, they faced every other team again. Results of the tournament satisfy the following condition: each team lost as many times as it won and there were no draws. Prove that some half of the matches can be ignored so that the condition will still hold.

(Magdaléna Mišinová)

Solution:
Let n be the number of teams in the tournament. We can represent the matches played during both tournament days as a directed multigraph. There will be an edge from team A to team B whenever A wins against B. That means that between every two vertices, there are precisely two edges. Let us fix an arbitrary vertex v. There are three types of these "double edges" incident with v. Both edges can be pointing to v. Both can be pointing away from v. Or they can be pointing in opposite directions. We will refer to the first two types as *same-edge* and the third as *anti-edge*. Let us denote by x, y and z, respectively, the number of these "double edges" incident with v. Because every team won as many times as it lost, it must be the case that $x = y$. But then the "double-degree" of v is $n - 1 = x + y + z = 2x + z$. Because n is odd, $n - 1$ is even, which means that z has to be even.

Next, we would like to select the matches to ignore. Notice that if we delete one edge from each same-edge, the condition that each team won as many matches as it lost will still be satisfied – this is because of the previous observation that $x = y$. Therefore, we would like to overlook one edge from each anti-edge in a way that the condition still holds. That way, we would ignore exactly half the matches.

Let us create a new, unoriented graph on the same set of vertices. We will put an edge between two teams whenever an anti-edge is in the original graph. The previous observation that z has to be even shows that each vertex has an even degree. A graph where this is the case is known as *Eulerian*. Therefore, there exists an Eulerian cycle in every connected component, that is a cycle which uses each edge exactly once. But then we can give an orientation to these cycles. And then, from each anti-edge, we forget the edge going in the opposite direction than the orientation of the cycle. This way, for each vertex, we keep the same number of incoming edges and outgoing edges, so the condition is satisfied and we are done.

Alternatively, if we are given a graph with even degrees, we can decompose it into cycles. We start at an arbitrary vertex and go along the edges until we revisit some vertex. Then we have a cycle from that vertex to itself. We can orient it in the same way as with the Eulerian cycle and then delete it. After the deletion, it still has to hold that every vertex has an even degree. Therefore, we can continue this way until all vertices have degree 0.

Poznámky:
Všechna správná řešení se ubírala velice podobnou úvahou jako to vzorové. Poměrně dost došlých řešení buď ne úplně správně pochopilo úlohu, nebo předpokládalo nějakou nepravdu. Jedna častější chyba byla, že se na situaci nemůžeme dívat izolovaně z jednoho vrcholu a argumentovat pouze paritou, protože pokud se rozhodneme ignorovat turnaj jednoho týmu, pak jej musíme ignorovat i pro jeho soupeře.

Dále se poměrně těžko rozhodovalo, jak moc jsou si jednotlivá řešení vědoma detailů v závěru důkazu – kde máme graf se všemi stupni sudými. Mnoho řešení opomnělo, že tento graf nemusí být souvislý. Také i eulerovský tah se dá vytvořit konstruktivně podobně jako dekompozice do cyklů.

Zde je ale důležité rozlišovat, zda při těchto konstrukcích uvažujeme o cyklech (nesmí se opakovat vrcholy ani hrany kromě počátečního) nebo o uzavřených tazích (nesmí se opakovat pouze hrany). Moc tomu nepomáhalo, že série byla anglická, tedy se lehce spletla terminologie *walk*, *path*, *trail*, *cycle* či *circuit*, a pak se těžko soudilo, zda je to chyba myšlenková, nebo jazyková.

(Lenka Kopfová)

Problem 7.

On Christmas Eve, n piglets exchanged some Christmas GIFs. We know that for each pair of piglets, at least one of them sent the other a Christmas GIF. We also know that every piglet received a Christmas GIF from exactly one quarter of the recipients of his Christmas GIFs. Determine all possible values of n. *(Marian Poljak)*

SOLUTION:

We can represent this problem as a directed graph on n vertices, where an edge from a to b means that piglet a sent a GIF to piglet b. If both a and b sent GIFs to each other, we will call this pair *friendly.* We will denote the number of friendly pairs as f.

Now let us count the number of edges. Each pair of vertices has to have at least one edge between them, and for every friendly pair, there will be exactly two edges. So the total number of edges is $\binom{n}{2}+f$. However, we can also count the number of edges in another way. Let us look at an arbitrary vertex and count the number of friendly pairs it is in – let's call this number r. Each friendly pair means one recipient of his GIF that also sent a GIF back. So from the problem statement, we know that this piglet has sent out exactly $4r$ GIFs. So the total number of sent GIFs is 4 times the sum of r's for every vertex, which is equal to twice the number of friendly pairs. This means that, in total, $4 \cdot 2 \cdot f = 8f$ GIFs were sent, and consequently there are $8f$ edges in our graph.

We have counted the number of edges in the graph in two different ways and these two results must coincide. Therefore, we get

$$\binom{n}{2} + f = 8f,$$

$$7f = \frac{n \cdot (n-1)}{2}.$$

From this equation, we see that n has to be either a multiple of seven or one more than a multiple of seven – because if this wasn't the case, then f wouldn't be an integer. All that is left to do is to prove that a suitable configuration exists for any n that can be written as $7k$ or $7k+1$ for a non-negative integer k. For $n = 0$ and $n = 1$, such configurations obviously exist. For all other values, we can use one of the following two constructions:

(1) $n = 7k$: Imagine that $7k-1$ piglets sit in a circle and each sends a GIF to the next $4k-1$ piglets in the clockwise direction and also to the one piglet who is not in the circle. This way, every piglet will receive a GIF from $4k-1$ piglets in the counter-clockwise direction, so there will be exactly $(4k-1)+(4k-1)-(7k-2) = k$ piglets which form a friendly pair with this piglet. That is a quarter of the number of GIFs that this piglet sent, so the condition from the task is met for every piglet in the circle. The one piglet that is not in the circle didn't send any GIFs, so the condition is also met for him. Hence this configuration is satisfactory.

(2) $n = 7k+1$: Now, imagine that all $7k+1$ piglets sit in a circle and let each of them send GIF to the next $4k$ piglet in the clockwise direction. This way, each piglet will receive $4k$ GIFs and will be a part of $4k+4k-7k = k$ friendly pairs, which is a quarter of the number of GIFs this piglet set. Since this holds for each piglet, so we have a valid configuration.

Therefore, we have shown that all values of n, which can be written as $7k$ or $7k+1$, are possible and all other values are not, so this is the end of our solution.

Poznámky:
Naprostá většina řešení správně odvodila, že hodnota n musí být ve tvaru $7k$, nebo $7k+1$. Nemálo řešení už ale pro tyto hodnoty neukázalo konstrukci a tedy nedokázalo, že všechny n v tomto tvaru jsou už vyhovující. Spousta řešení také za možné hodnoty n neurčila nulu a jedničku, které ale zadání vyhovují (i když je v těchto případech posláno nula GIFů). Za to jsem však body nestrhával.

(Martin Fof)

Problem 8.

In the kingdom of Esarpia, there live $n \geq 3$ peasants. Some pairs of these peasants are friends[2]. We say that four peasants a, b, c and d form a friendly square, if the four pairs $\{a,b\}$, $\{b,c\}$, $\{c,d\}$ and $\{d,a\}$ are friends, but the pairs $\{a,c\}$ and $\{b,d\}$ are not.

King Esarp enacted a decree that a several day long festival is to be held to celebrate the birthday of his most trusted Archmage. The decree stipulates the following rules:

- *On each day of the festival, a party must be held. Once no more parties can be held, the festival ends.*
- *A group of peasants may hold a party, only if they are all friends with each other and no peasant that does not participate is friends with all the participants.*
- *No two parties can have precisely the same set of participants.*

Prove that if there are no friendly squares in Esarpia, then the festival can last at most $\frac{n(n-1)}{2}$ days.

(Magdaléna Mišinová)

Solution:
First, let's formalize the problem using graphs: let each peasant be represented by a vertex and draw an edge between two vertices if and only if the corresponding peasants are friends. Denote this graph G. We are to prove that if G does not contain any friendly squares, then there are at most $\frac{n(n-1)}{2}$ different *maximal cliques* in G (maximal in the sense that there is no larger clique containing it). We will solve the problem by using mathematical induction.

For the base case, let's start with $n = 3$. There are precisely 4 different graphs on 3 vertices up to symmetry. If there are no edges in G, we have exactly 3 single element maximal cliques. If there is exactly one edge, we have two maximal cliques (the two vertices of the edge and the remaining vertex alone). If there are exactly two edges, there are also only two maximal cliques (the two vertices of each edge). Finally, if G is complete, then we have only one maximal clique (consisting of all vertices). Overall, we have at most $3 = \frac{3\cdot 2}{2}$ maximal cliques as we wanted.

For the induction step, suppose the statement holds for all graphs on $n-1$ vertices. We will prove it holds for n as well. Pick a vertex v of G, denote by $C_1, \ldots, C_k$ the maximal cliques containing v and $D_1, \ldots, D_l$ the maximal cliques not containing v.

If v has no neighbours, then $k = 1$, as the only maximal clique containing v is $C_1 = \{v\}$. Since $D_1, \ldots, D_l$ are also maximal cliques in the graph $G \setminus \{v\}$, we have $l \leq \frac{(n-1)(n-2)}{2}$ by induction hypothesis. Hence

$$k + l \leq 1 + \frac{(n-1)(n-2)}{2} < \frac{n(n-1)}{2}$$

and we are done.

If v has at least one neighbour, then the cliques $C_1, \ldots, C_k$ must each contain at least 2 vertices. For $1 \leq i \leq k$ denote $C_i' = C_i \setminus \{v\}$. These can be viewed as cliques in the graph $G \setminus \{v\}$. They are different from $D_1, \ldots, D_l$, since D_i's are maximal in G. Denote m the number of cliques in $C_1', \ldots, C_k'$ that are not maximal in $G \setminus \{v\}$. By induction hypothesis, we get

$$(k-m) + l \leq \frac{(n-1)(n-2)}{2}.$$

[2]Friendship is symmetric.

Note that if we now show that m, the number of non-maximal cliques C_i', is at most

$$n-1=\frac{n(n-1)}{2}-\frac{(n-1)(n-2)}{2},$$

we will be done.

Let's consider some clique C_i such that C_i' is not maximal in $G\setminus\{v\}$. Its non-maximality implies there exists an *extending vertex* $u \in G \setminus C_i \subset G \setminus \{v\}$ such that all elements of C_i' are friends with u but u is not friends with v (otherwise $C_i \cup \{u\}$ would be a clique contradicting the maximality of C_i). For the sake of contradiction, suppose that $m > n-1$. Then we can find two non-maximal cliques C_i', C_j' with the same extending vertex u. Now, since C_i and C_j are different, there must exist two vertices $x_i \in C_i'$, $x_j \in C_j'$ that are not connected by an edge (otherwise $C_i \cup C_j$ would be maximal clique). However, this means that the vertices v, x_i, u, x_j form a friendly square, which contradicts the conditions of the problem. We have, therefore, completed the induction step.

Poznámky:
Všechna správná řešení použila stejný argument indukcí jako vzorové řešení. Pro zajímavost, pokud bychom ze zadání vypustili podmínku na bezčtvercovost daných grafů, tak existují grafy s více než $\frac{n(n-1)}{2}$ maximálními klikami ($n = 12$ je nejmenší n, pro které takový graf existuje, přičemž obecně je maximální možný počet exponenciálně závislý na n). *(Martin Raška)*