Polynomials

4TH AUTUMN SERIES — DATE DUE: 10TH JANUARY 2022

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
Daniel would like to have two polynomials $P(x)$, $Q(x)$ such that the degree of the product $P(x)\cdot Q(x)$ is six and the degree of the sum $P(x)+Q(x)$ is two. Help him find an example of such polynomials.

PROBLEM 2. (3 POINTS)
Klátra owns the polynomial $P(x) = x^2 + 8x + 12$. Prove that for any positive integer n the value $P(n)$ is not a prime number.

PROBLEM 3. (3 POINTS)
Fíla gave Áďa three integers a, b, c for her birthday. Martin gave her a polynomial $P(x)$ with integer coefficients satisfying $P(a) = 1$, $P(b) = 2$ and $P(c) = 3$. Prove that b lies between a and c.

PROBLEM 4. (5 POINTS)
Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be a polynomial with roots $x_1, \ldots, x_n$. Express $(x_1^2 - 1)\cdots(x_n^2 - 1)$ in terms of $a_0, a_1, \ldots, a_{n-1}$.

PROBLEM 5. (5 POINTS)
Vašek hid his favourite n-tuple $a_1, \ldots, a_n$ of real numbers in a vault and the code is

$$a_1^2 + a_2^2 + \cdots + a_n^2.$$

Majda would like to steal his n-tuple, but she only knows that it satisfies

$$1 + x^n + x^{2n} = (1 + a_1x + x^2)\cdot(1 + a_2x + x^2)\cdots(1 + a_nx + x^2)$$

for all real x. Help Majda find the code in terms of n.

PROBLEM 6. (5 POINTS)
Find all non-constant polynomials P, Q with real coefficients which satisfy

$$P\left(Q(x)^3\right) = x\cdot P(x)\cdot Q(x)^3$$

for all real x.

PROBLEM 7. (5 POINTS)
Tajl lost all his polynomials. He only knows that his polynomials were exactly those polynomials $P(n)$ with integer coefficients which satisfy

$$P(n) \mid n! + 2$$

for all positive integers n. Find all Tajl's polynomials.

PROBLEM 8. (5 POINTS)

Ducky is swimming in a pond full of all polynomials. Polynomial $P(x)$ is called *fishy* if all of its coefficients are integers and there are infinitely many pairs (a, b) of coprime[1] positive integers, which satisfy

$$a + b \mid P(a) + P(b).$$

Find all fishy polynomials.

[1] Two numbers are called coprime if their greatest common divisor is one.