Distances

4^{TH} AUTUMN SERIES DATE DUE: 11^{TH} JANUARY 2021

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
There is a piglet standing inside a square field of size 10×10. In each corner of the field, there is a goat observing the piglet. The piglet's distance to some two of the goats is 6 and 8, respectively. Find its distance to the remaining two goats.

PROBLEM 2. (3 POINTS)
There is an odd number (at least three) of politicians standing on a meadow so that no two pairs of politicians are the same distance apart. Each politician has an egg. Suddenly, each of them throws his egg and hits the face of the politician standing closest to him. Show that afterwards, there is a politician that does not have egg on his face.

PROBLEM 3. (3 POINTS)
In the plane, n points are coloured red in such a way that the distance between any two of them is at least 1. Prove that there are at most $3n$ pairs of red points whose distance is exactly 1.

PROBLEM 4. (5 POINTS)
A quadrilateral $ABCD$ is inscribed in a circle ω with centre O in such a way that the diagonals AC and BD are perpendicular. Prove that the distance from O to the line CD is $\frac{1}{2}|AB|$.

PROBLEM 5. (5 POINTS)
Let $ABCD$ be a cyclic quadrilateral. Let P denote the intersection of its diagonals and O its circumcentre. Finally, let K, L, M, N be the circumcentres of $\triangle AOP$, $\triangle BOP$, $\triangle COP$, $\triangle DOP$. Prove that $KL = MN$.

PROBLEM 6. (5 POINTS)
Given a regular 100-gon with 10 blue, 10 red, and 80 colourless vertices, prove that there must be a pair of blue vertices with the same distance as some pair of red vertices.

PROBLEM 7. (5 POINTS)
Let ABC be a triangle and BD, CE its altitudes. Let G be a point on BD such that $GE = DE$. Let ℓ denote the line through C parallel to GE. The line through G parallel to AC intersects ℓ at H. Prove that $BH = BC$.

PROBLEM 8. (5 POINTS)
Let S be a set of n points in the plane such that no four points lie in one line. Let $\{d_1, d_2, \cdots, d_k\}$ be the set of distances between pairs of distinct points in S, and let m_i be the multiplicity of d_i, i.e. the number of unordered pairs of points whose distance is d_i. Prove that

$$m_1^2 + m_2^2 + \cdots + m_k^2 \leq n^3 - n^2.$$