Introductory Text to the 4th Autumn Series

The topic of this year's 4^{th} autumn series is distances. This text should serve you as an introduction to the terminology as well as an overview of some methods.

Let's consider two points in the plane, $A = (x_a, y_a)$, $B = (x_b, y_b)$. The distance between A and B is usually denoted simply by AB contrary to the notation $|AB|$ you might be familiar with from your high school. It can be computed by the formula below:

$$AB = \sqrt{(x_a - x_b)^2 + (y_a - y_b)^2}.$$

The set of points whose distances from A and B are equal is called a *perpendicular bisector* of the segment $\overline{AB}$. Given a polygon $\mathcal{P}$, if there exists a point O such that O has the same distance to each vertex of $\mathcal{P}$, we call $\mathcal{P}$ *cyclic* and O is said to be the *circumcenter* of $\mathcal{P}$. In this scenario, O is the intersection of perpendicular bisectors of all of the sides of $\mathcal{P}$.

We can also speak about distances between points and lines – for a point X and a line ℓ, we define the distance between X and ℓ as the length of the line segment beginning at X, ending at ℓ and perpendicular to ℓ.

We will finish by solving some problems. The first one is an easy geometry illustrating the connection between distances and areas.

Problem. Let $ABCD$ be a trapezoid where $AB \parallel CD$. Let E, F be points on BC, DA respectively such that $AE \perp BC$ and $BF \perp DA$. Suppose that $AE = BF$ holds. Prove that $BC = DA$.

First, suppose that lines BC and DA meet at X. WLOG[1] we can assume that X, D, A lie on the line in this particular order. By expressing the area of a triangle XAB in two ways and using the condition $AE = BF$, we obtain

$$\frac{AE \cdot BX}{2} = S_{\triangle XAB} = \frac{BF \cdot XA}{2},$$
$$AX = BX.$$

Moreover, since triangles XAB and XDC are similar, we have $XC = XD$. Therefore $BC = XB - XC = XA - XD = DA$ as desired.

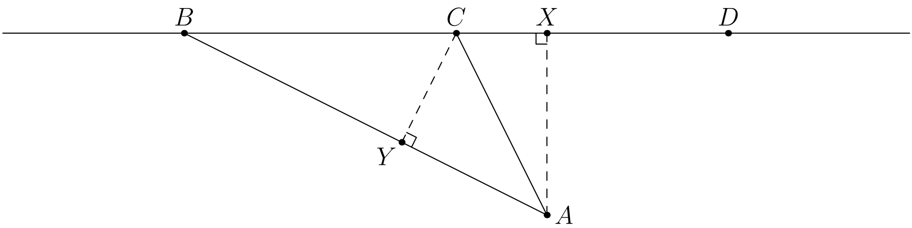
On the other hand, if the lines BC and DA are parallel, then $ABCD$ is a parallelogram, where $BC = AD$ surely holds. □

We will finish by proving a very intuitively looking theorem which stumped the world's best mathematicians for fifty years. The proof, albeit easy to understand, finds an ingenious minimum distance argument where there is seemingly no natural way to use them.

[1] This is a very common acronym which stands for "Without loss of generality".

Theorem. (Sylvester-Gallai) Given a set $\mathcal{S}$ consisting of finitely many points (at least two) in the plane, not all of which are collinear, there exists a line which passes through exactly two points in $\mathcal{S}$.

Because $\mathcal{S}$ is finite and the points are not all collinear, there must be an ordered triple (A, B, C) of points from $\mathcal{S}$ such that the distance between A and BC is minimal among all such triples. We will prove by contradiction that the line BC does not pass through any other point from $\mathcal{S}$.



Suppose that $B, C, D \in \mathcal{S}$ are collinear. Moreover, let X be a point on BC such that $AX \perp BC$. Then at least two points from $\{B, C, D\}$ lie on the same side of X, so WLOG $|\sphericalangle ACB| \geq 90°$. Finally, we denote the foot of the altitude from C in the triangle ABC by Y. By expressing the area of a triangle ABC in two ways we obtain

$$\frac{AB \cdot YC}{2} = \frac{BC \cdot XA}{2} < \frac{AB \cdot XA}{2}.$$

Therefore $0 < YC < XA$ which is a contradiction with the minimality of XA. □

Good luck with solving the problems from 4^{th} autumn series.