Introductory Text to the 4th Autumn Series

The topic of the 4th autumn series this year is Circles. This text should provide you with a basic insight into the theorems you might want to use together with a vocabulary which will be useful when writing down your solutions. You may use the theorems stated here without proving them.

Angles in Circles

In English literature, we often denote a circle by ω and its centre by O. If A and B are two points on the circle, then the line segment AB is called a *chord*. Also if X is a point on the same circle, then we say the angle AXB is *subtended* by the chord AB, or equivalently by the arc AB.

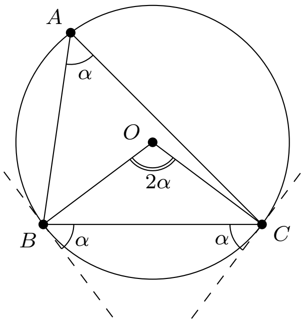
Now let C be another point on ω. Then the angle BAC is said to be *inscribed* in ω and the angle BOC is said to be a *central angle* in ω.

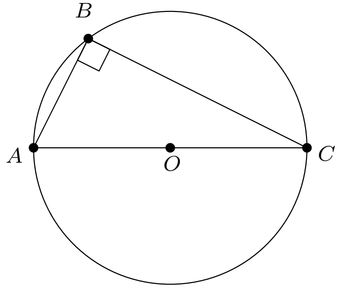
These angles play an important role in the two upcoming theorems, which are quite short, but very powerful and useful in many problems and theorems in geometry.

Theorem. (Inscribed Angle Theorem) An angle inscribed into a circle is half of the central angle subtended by the same chord.

Corollary. (Thales's Theorem) If A, B, C are distinct points on a circle where AC is the diameter, then ABC is a right angle.

Theorem. (Tangent Chord Theorem) An angle between a tangent line and a chord is half of the central angle subtended by the same chord.





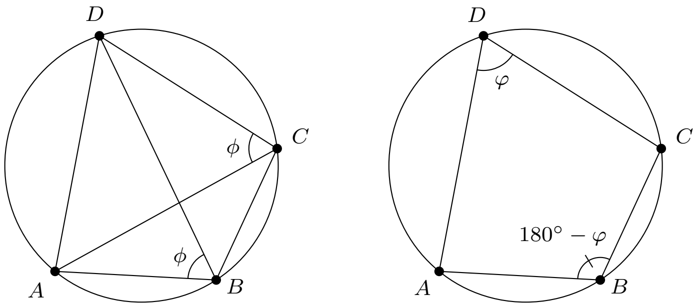
Cyclic Quadrilateral

We say a set of points is *concyclic* if they lie on a common circle, i.e. if they are all equally distant from a single centre. A quadrilateral whose vertices are concyclic is then called a *cyclic quadrilateral.*

Using the two theorems above, we can notice a few interesting properties of cyclic quadrilaterals:

Proposition. A quadrilateral $ABCD$ is cyclic if and only if

(1) the angle between a side and a diagonal is equal to the angle between the opposite side and the other diagonal, for example $\angle ABD = \angle ACD$,
(2) the opposite angles sum to $180°$, for example $\angle ABC + \angle ADC = 180°$.



Cyclic quadrilaterals reveal a lot of information about angles we are interested in, so it is worth trying to find them and using the information they give us. So let's try it out on an example:

Problem. Let $ABCD$ be a square and let X be a point on the shorter arc AB of its circumcircle. Let Y be the intersection of XC with AB and Z the intersection of XD with AC. Prove that YZ is perpendicular to AC.

Solution. Suppose $BCZY$ is a cyclic quadrilateral. Then by applying part (2) of the above proposition, we have $\angle YBC + \angle YZC = 180°$. And as YBC is a right angle, so is YZC. Now we just need to show $BCZY$ is cyclic. Both angles ADX and ACX are inscribed to the circumcircle of $ABCD$ and are subtended by the same arc AX, so by the Inscribed Angle Theorem they are congruent. Furthermore, from the symmetry of B and D with respect to AC, we have $\angle ADZ = \angle ABZ$. Hence

$$\angle ZCY = \angle ACX = \angle ADX = \angle ADZ = \angle ABZ = \angle ZBY$$

and by part (1) of the above proposition, $BCZY$ is cyclic, as desired.

Hopefully this text gave you a decent start into the next series. If you think you struggled to understand this text, or if you would like to know more about this topic, we have a great introductory text (written in Czech) from last year. You can find it at *https://mks.mff.cuni.cz/archive/38/uvod2j.pdf*. Otherwise, happy solving!