Circles

4. podzimní série Termín odeslání: 6. ledna 2020

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

Úloha 1. (3 body)
Kua would like to draw 7 circles on a piece of paper so that every two circles intersect at two distinct points. And he would love it if there were exactly 7 points, in which 2 or more circles intersect. Show him how this is possible.

Úloha 2. (3 body)
Let ω_1, ω_2 be two circles with centres O_1, O_2 intersecting at points X, Y such that $\angle O_1XO_2 = 90°$. Let D be the intersection of O_1O_2 and ω_1 such that O_1 lies between D and O_2. Let P be the intersection of DX and ω_2 distinct from X. Prove that PO_2 is perpendicular to O_1O_2.

Úloha 3. (3 body)
Jáchym drew three circles on a whiteboard. The circles had radii 2, 3, and 3 and each two were externally tangent. Then he drew the circle ω that is internally tangent to all three of them. Help him calculate the radius of ω.

Úloha 4. (5 bodů)
Let $ABCD$ be a cyclic quadrilateral with circumcentre O such that AC and BD are perpendicular. Let ω_1, ω_2, ω_3, and ω_4 be circles, where the diameters of these circles are AO, BO, CO, and DO respectively. Finally, let P, Q, R, and S be the intersections of ω_1 with ω_2, ω_2 with ω_3, ω_3 with ω_4, and ω_4 with ω_1 respectively, distinct from O. Prove that $PQRS$ is a rectangle.

Úloha 5. (5 bodů)
Let ω_1 and ω_2 be two circles externally tangent at T. Let C be a point on ω_2 such that the tangent at C intersects ω_1 at two distinct points X and Y. Now define P as the intersection of CT and ω_1 distinct from T. Show that PXY is an isosceles triangle.

Úloha 6. (5 bodů)
There are $2n$ points on a circle labeled $1, 2, \dots, 2n$ in some order. We define a *pairing* as a set of n segments between these points such that every point is an endpoint of exactly one of the segments. For a segment connecting points labelled a and b, we say its *value* is the number $|a-b|$. Finally, we say a pairing is *good*, if the sum of values of all n segments is equal to n^2. Show that for any initial order of labels there exists a good pairing such that no two segments intersect.

Úloha 7. (5 bodů)
Let $ABCD$ be a parallelogram such that $\angle DAB$ is obtuse. Then, let M be the midpoint of AB and E be the intersection of the circumcircle of DAB and the line DM distinct from D. Finally, let H be the point on DA such that $\angle AHB = 90°$. Prove that C, D, H, and E are concyclic.

Úloha 8. (5 bodů)
Let ABC be a non-equilateral triangle and G its centroid. Denote the midpoints of line segments AB, CA, BC, AG, CG, and BG by M_C, M_B, M_A, N_A, N_C, and N_B respectively. Show that the circumcircles of $M_CN_AM_B$, $M_BN_CM_A$, and $M_AN_BM_C$ all intersect in a single point.

Circles

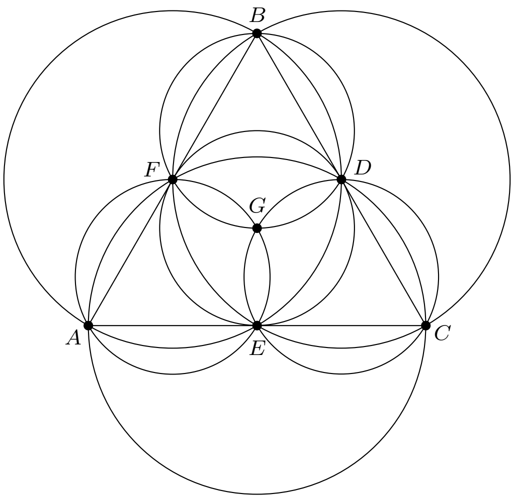
4^{TH} AUTUMN SERIES — MODEL SOLUTIONS

Problem 1.

Kua would like to draw 7 circles on a piece of paper so that every two circles intersect at two distinct points. And he would love it if there were exactly 7 points, in which 2 or more circles intersect. Show him how this is possible.

(Filip Bialas)

SOLUTION:
We can draw an equilateral triangle with its centre and midpoints of its sides. And then draw circles as in the diagram.



More precisely, denote the vertices of the triangle by A, B, C, midpoints of the opposite sides by D, E, F and its centre by G. Then it suffices to consider circles with diameters AG, BG, CG, AB, BC, AC and circumcircle of DEF.

POZNÁMKY:
Většina řešitelů přišla s podobným obrázkem jako ve vzoráku. Pouze dva řešitelé našli mnohem méně symetrický obrázek, kde jedním z vrcholů procházelo hned šest kružnic. U všech ostatních, kteří tuhle úlohu zdárně vyřešili, šlo o kombinatoricky stejnou konstrukci – průsečíky šlo vždy pojmenovat tak, aby na nakreslených sedmi kružnicích ležely stejně pojmenované body jako na těch ve vzorovém řešení. I když u některých vypadal obrázek na první pohled o dost jinak.

Matoušovi Šafránkovi jsem udělil $+i$ za to, že si uvědomil, že při nalézání obrázku nemusí používat pouze kružnice, ale také přímky. Vzniklý obrázek pak již jen stačí zinvertovat podle libovolné kružnice se středem v bodě, který neleží na žádné z nakreslených přímek a kružnic. Pokud poslední větě nerozumíte, ale chtěli byste jí rozumět, stačí se podívat do druhého dílu letošního seriálu. *(Filip Bialas)*

Problem 2.

Let ω_1, ω_2 be two circles with centres O_1, O_2 intersecting at points X, Y such that $\angle O_1XO_2 = 90°$. Let D be the intersection of O_1O_2 and ω_1 such that O_1 lies between D and O_2. Let P be the intersection of DX and ω_2 distinct from X. Prove that PO_2 is perpendicular to O_1O_2.

(Pavel Hudec)

SOLUTION:

Since DO_1 and XO_1 are radii of ω_1, traingle DO_1X is isosceles. Using a similar argument, we get that $PO_2 = XO_2$, so triangle PO_1X is also isosceles. Hence we have

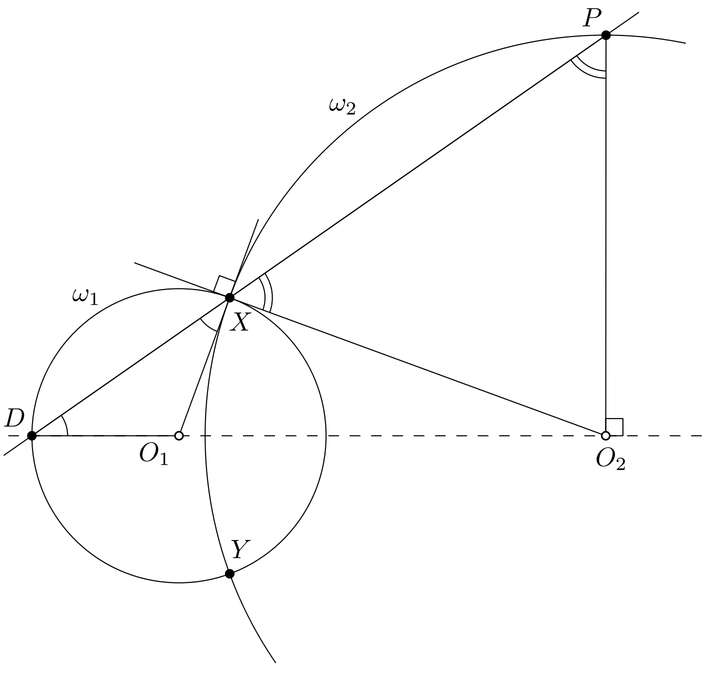
$$\angle XDO_1 = \angle DXO_1,$$
$$\angle XPO_2 = \angle PXO_2.$$

The points D, X, P are colinear, therefore

$$\angle DXO_1 + \angle O_1XO_2 + \angle O_2XP = 180°,$$
$$\angle DXO_1 + \angle O_2XP = 90°,$$
$$\angle XDO_1 + \angle O_2PX = 90°.$$

Because the sum of internal angles in the triangle DPO_2 is $180°$, we can get the desired conclusion:

$$\angle DO_2P = 180° - \angle O_2DX - \angle O_2PX = 90°.$$



Poznámky:
Úlohu vyřešila většina řešitelů správně. Jen připomínám, že řešení má být pochopitelné i bez přiloženého obrázku, tedy je potřeba v textu definovat i všechny úhly, které nějak označíme, a provést všechny výpočty, ne se pouze odkázat na obrázek. *(„madam Verča“ Hladíková)*

Problem 3.

Jáchym drew three circles on a whiteboard. The circles had radii 2, 3, and 3 and each two were externally tangent. Then he drew the circle ω that is internally tangent to all three of them. Help him calculate the radius of ω.

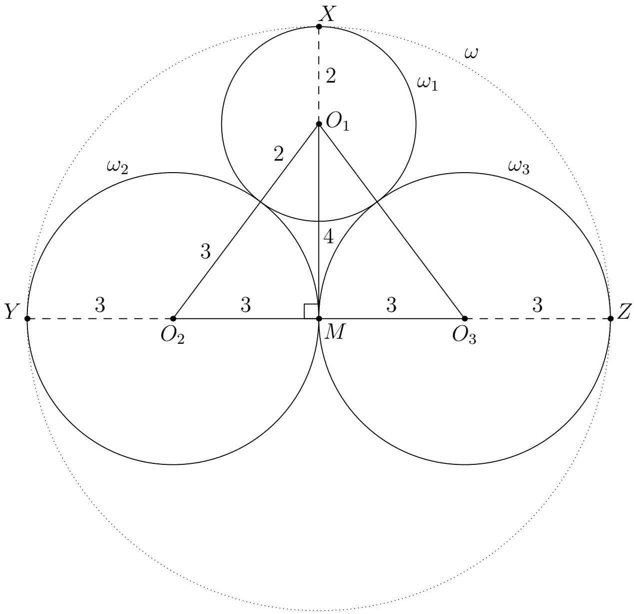
(Lucien Šíma)

Solution:
Let ω_1, ω_2, ω_3 be the three circles and O_1, O_2, O_3 their respective centres (ω_1 has a radius of 2). Let M be the midpoint of O_2O_3. The configuration is symmetric with respect to the common tangent line of ω_2 and ω_3, namely the line O_1M, therefore O_1M is perpendicular to O_2O_3 and the centre of ω lies on O_1M. The length of O_1M can be obtained from the Pythagorean theorem as

$$O_1M = \sqrt{O_1O_2^2 - O_2M^2} = \sqrt{5^2 - 3^2} = 4.$$

Let X be the intersection of O_1M with ω_1 that is further away from M and let Y and Z be the intersections of O_2O_3 with ω_2 and ω_3 respectively distinct from M. Then the centre of ω_1 lies on the line XM and $XM = XO_1 + O_1M = 2 + 4$. Similarly, O_2 and O_3 lie on the line MY (and MZ) and $MY = MZ = 3 + 3 = 6$. Therefore if we draw a circle with the centre M and radius 6, it will be internally tangent to all three circles at points X, Y, and Z.

The radius of ω is 6.



Poznámky:
Většina si s úlohou hravě poradila, ale několik řešitelů místo poloměru kružnice, která má se zadanými kružnicemi vnitřní dotyk (*internally tangent*), spočítalo poloměr kružnice, která má dotyk vnější (*externally tangent*), a tím si vysloužili nula bodů. *(Hedvika Ranošová)*

Problem 4.

Let $ABCD$ be a cyclic quadrilateral with circumcentre O such that AC and BD are perpendicular. Let ω_1, ω_2, ω_3, and ω_4 be circles, where the diameters of these circles are AO, BO, CO, and DO respectively. Finally, let P, Q, R, and S be the intersections of ω_1 with ω_2, ω_2 with ω_3, ω_3 with ω_4, and ω_4 with ω_1 respectively, distinct from O. Prove that $PQRS$ is a rectangle.

(Jáchym Solecký)

Solution:
First, we would like to prove that P, Q, R, S are the midpoints of segments AB, BC, CD, DA respectively. Have a look at point P which lies on two circles with diameters AO and BO. Therefore, from Thales' theorem, we know that the angles APO and BPO are right. So the size of the angle APB is

$$\angle APB = \angle AOP + \angle BOP = 180^\circ.$$

This means that P lies on the segment AB. In addition, triangle AOB is isosceles with the base AB (because AO and BO are radii of the circumcircle of the quadrilateral $ABCD$). So the altitude from point O to the base is also a median and therefore P is the midpoint of AB. By analogy, we get the same result for the rest of the points Q, R, S.

As a result of that, we know that the lines PQ, QR, RS, SP are the midlines of the triangles ABC, BCD, CDA, DAB respectively.

Then, as we know, in triangle KLM the midline connecting the midpoints of KL and KM is parallel to the third side LM. This implies that PQ and SR are parallel to their base AC, analogically QR and SP are parallel to their base BD. Moreover, AC is perpendicular to BD and this all together means that PQ and SR are perpendicular to QR and SP, so $PQRS$ is a rectangle.

Poznámky:
Většina došlých řešení dostala plný počet bodů. Někteří řešitelé považovali některé věci za zřejmé, třeba že body P, Q, R, S leží na úsečkách AB, BC, CD, DA, ačkoli by zasloužily důkaz. Proto jim byl občas strhnut bod. Jinak bylo všechno v pořádku a neměl jsem žádné velké výtky.

(Fíla Čermák)

Problem 5.

Let ω_1 and ω_2 be two circles externally tangent at T. Let C be a point on ω_2 such that the tangent at C intersects ω_1 at two distinct points X and Y. Now define P as the intersection of CT and ω_1 distinct from T. Show that PXY is an isosceles triangle.

(Pavel Hudec)

Solution:
Let O_1, O_2 be the centres of ω_1, ω_2. Consider the homothety with center T which sends ω_1 to ω_2. The image of the line O_1P is O_2C, therefore $O_1P \parallel O_2C$ (since in a homothety, every line is parallel to its image). It follows that $O_1P \perp XY$ because XY is tangent to ω_2 and perpendicular to O_2C. Finally, O_1 lies on the perpendicular bisector of XY, hence O_1P and the perpendicular bisector coincide. We have proven that P lies on the perpendicular bisector of XY which implies $PX = PY$ and so PXY is isosceles.

Poznámky:
Téměř všechna řešení byla správná a většina z nich postupovala podobně jako vzorové řešení. Ukázalo se, že úlohu je možné řešit mnoha různými způsoby. Část řešitelů například využívala průsečíku XY a společné tečny ke kružnicím v bodě T. *(Josef Minařík)*

Problem 6.

There are $2n$ points on a circle labelled $1, 2, \dots, 2n$ in some order. We define a pairing as a set of n segments between these points such that every point is an endpoint of exactly one of the segments. For a segment connecting points labelled a and b, we say its value is the number $|a-b|$. Finally, we say a pairing is good, if the sum of values of all n segments is equal to n^2. Show that for any initial order of labels there exists a good pairing such that no two segments intersect.

(Pavel Hudec)

Solution:

First, colour points labelled $1, 2, \dots, n$ as red and points labelled $n+1, n+2, \dots, 2n$ as blue. We will say a pairing is *codenamesish* if every segment connects a red point with a blue point and no two segments intersect. Assume that we have a codenamesish pairing. Then any blue point (labelled by a) would contribute to the sum of values of all n segments by a, whereas any red point (labelled by b) would contribute by $-b$. Therefore the sum of values is equal to

$$\Big((n+1)+(n+2)+\cdots+2n\Big)-(1+2+\cdots+n)=n^2.$$

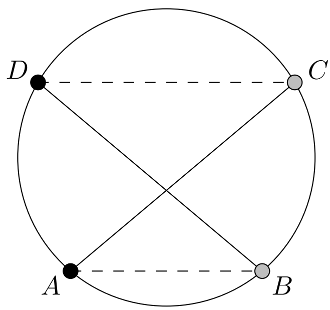
We will present two proofs of existence of a codenamesish pairing.

First solution (connecting neighboring points):

Use induction on n. The base case $n=1$ is clear. Now suppose there are n blue and n red points on the circle. Note that there exist two neighboring points of opposite colours. Connect them by a new segment. By the induction hypothesis there exists a codenamesish pairing of the remaining points. These all lie in one half-plane defined by the new segment, therefore no other segment can intersect the new segment. We can then conclude that the constructed pairing is indeed codenamesish.

Second solution (reducing the number of intersections):

Consider a pairing where every segment connects a red point with a blue point with the smallest number of intersecting pairs of segments. For the sake of contradiction, suppose that there exist two intersecting segments AC and BD. Now, connect the points A, B, C, D in the other way such that both segments still connect a red point with a blue point. WLOG let the connected pairs of points now be AB and CD.



Notice that AB and CD don't intersect, which leaves us with one fewer intersection points. If any other segment intersects both AB and CD, then it has to intersect both AC and BD, as the quadrilateral $ABCD$ is convex. If it intersects just one of them, say AB, it has to intersect at least one of the diagonals as well, since AB and parts of AC and BD form a triangle. Therefore the number of intersections after the swap decreased by at least one. This contradicts the minimality property of the pairing.

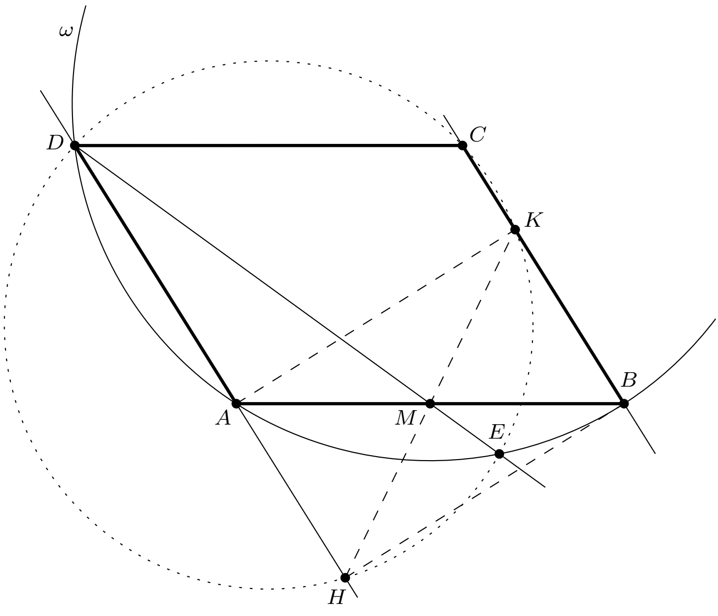
Poznámky:
Úloha dopadla velmi dobře, pouze několik málo řešitelů nad ní zaváhalo. Při konstrukci hledaného párování řešitelé využívali množství různých přístupů. Většina řešení byla podobná prvnímu vzorovému, ovšem byla k vidění i řešení používající diskrétní spojitost nebo některý z možných extremálních principů. *(Pavel Hudec)*

Problem 7.

Let $ABCD$ be a parallelogram such that $\angle DAB$ is obtuse. Then, let M be the midpoint of AB and E be the intersection of the circumcircle of DAB and the line DM distinct from D. Finally, let H be the point on DA such that $\angle AHB = 90°$. Prove that C, D, H, and E are concyclic.

(Matěj Doležálek)

Solution:
Let ω be the circumcircle of DAB and let K be the point on BC such that $\angle AKB = 90°$. Then $AHBK$ is a rectangle since AH and BK are parallel and two opposite angles are right.



We shall prove that D, C, K, H are concyclic and that D, K, E, H are concyclic. That will mean that all four points C, D, H, E lie on the circumcircle of DKH, solving then the problem.

First, the diagonals of a rectangle have the same length, so $HK = AB$. Since DH and CK are parallel, this means that $DCKH$ is an isosceles trapezoid and thus cyclic. Second, because the diagonals of a rectangle halve each other, M is the midpoint of both AB and HK, so the power[1] of M with respect to ω gives us

$$DM \cdot ME = AM \cdot MB = HM \cdot MK.$$

This means that D, K, E, H are concyclic, just as we wished to prove.

[1] If you are not familiar with the power of a point, you can learn more in this handout (written in Czech): *https://prase.cz/library/MocnostbodukekruzniciAL/MocnostbodukekruzniciAL.pdf*.

Poznámky:

Valná většina došlých řešení obdržela plný počet bodů. Z těchto úspěšných řešení valná většina nějakým způsobem využívala dokreslení bodu K, přičemž mnohá se namísto mocnosti k dokazované koncyklicitě dostala pouze úhlením. Při úhlení je často třeba dát si pozor na to, že zadání umožňuje více různých konfigurací bodů (v jakém pořadí leží na kružnici či na přímce), což vede k trochu odlišnému, ač v principu pořád stejnému úhlení. Opomenutí jedné nebo více takových konfigurací je jedním z nejspolehlivějších způsobů, jak např. v matematické olympiádě zbytečně ztratit body (ač v této úloze jsem za to nakonec body nestrhával). Způsobem, jak se rozebírání konfigurací vyvarovat, je *orientované úhlení*[2]. Lze si všimnout, že vzorové řešení tímto problémem netrpí, neboť lichoběžník má stejně dlouhá ramena, právě pokud má stejně dlouhé úhlopříčky (nezáleží na tom, jestli jsou DC a HK ramena, nebo úhlopříčky), a mocnost na konfiguraci nezávisí.

Jediným dalším funkčním dokreslením, které se v došlých řešeních vyskytlo, bylo využití bodu X, který je průsečíkem přímky DC s kružnicí opsanou DAB různým od D. *(Matěj Doležálek)*

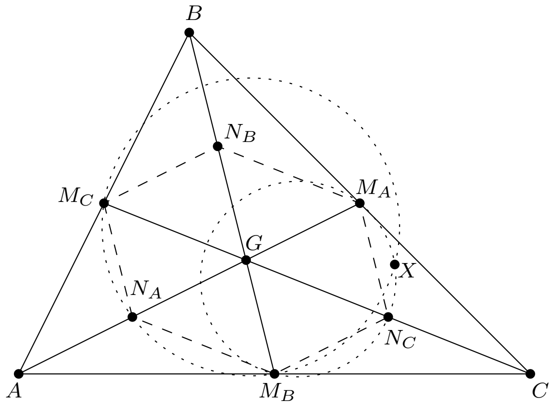
Problem 8.

Let ABC be a non-equilateral triangle and G its centroid. Denote the midpoints of line segments AB, CA, BC, AG, CG, and BG by M_C, M_B, M_A, N_A, N_C, and N_B respectively. Show that the circumcircles of $M_CN_AM_B$, $M_BN_CM_A$, and $M_AN_BM_C$ all intersect in a single point.

(Radek Olšák)

Solution:

Since M_AN_C is a midline of triangle BGC, it is parallel to BG. And as B, N_B, G and M_B lie on one line, it is also parallel to N_BM_B. Similarly, N_AM_C is parallel to N_BM_B, so we have $N_AM_C \parallel M_AN_C$. Analogously $N_BM_A \parallel M_BN_A$ and $N_CM_B \parallel M_CN_B$.



Now we will use directed angles[3]. Let X be the intersection of circumcircles of $M_BN_AM_C$ and $M_BN_CM_A$ distinct from M_B. Then

$$\begin{aligned}\angle(M_AX, XM_C) &= \angle(M_AX, XM_B) + \angle(M_BX, XM_C)\\ &= \angle(M_AN_C, N_CM_B) + \angle(M_BN_A, N_AM_C)\\ &= \angle(N_BM_B, M_CN_B) + \angle(N_BM_A, M_BN_B)\\ &= \angle(M_AN_B, N_BM_C),\end{aligned}$$

[2]Viz např. *https://prase.cz/library/OrientovaneUhleniMO/OrientovaneUhleniMO.pdf*.

[3]Accessible from *https://prase.cz/library/OrientovaneUhlyMTa/OrientovaneUhlyMTa.pdf*.

where the second equality follows by concyclicity of M_B, N_A, M_C, X and M_B, N_C, M_A, X, and the third equality follows since $M_A N_C \parallel N_B M_B$, $N_A M_C \parallel N_B M_B$, $N_B M_A \parallel M_B N_A$, and $N_C M_B \parallel M_C N_B$. That means that M_A, N_B, M_C, and X are concyclic, which is what we wanted.

Poznámky:
Většina přijatých řešení byla správně. Řešitelé si více či méně zdatně poradili s tím „jak vypadá obrázek“, přičemž nejjednodušší bylo postupovat přes orientované úhly, ale dalo se samozřejmě i jinak. *(Rado van Švarc)*