Equations

4^{TH} AUTUMN SERIES   DATE DUE: 7^{TH} JANUARY 2019

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
Anička wrote $1 \square 2 \square 3 \square 4 \square 5 = 5$ on a blackboard. Then E.T. came along and wrote either a plus sign or a minus sign (of his choice) into one of the squares on the left side of the equation. Could E.T., being a pest, make it so that Anička cannot make the equation hold by filling the remaining three gaps with plus or minus signs?

PROBLEM 2. (3 POINTS)
Solve for $x \in \mathbb{R}$:

$$|x| + |x+1| + \cdots + |x+2018| = x^2 + 2018x - 2019.$$

PROBLEM 3. (3 POINTS)
Pavel and Hedvika went for a walk by the seashore and found five cute baby seals. Because Pavel is a proper mathematician, he assigned a positive real number to each seal in order to quantify their cuteness. Then Hedvika noticed, that the product of *cutenesses* of any pair of baby seals is equal to the sum of *cutenesses* of the other three seals. What are the possible values of *cutenesses* of the five baby seals?

PROBLEM 4. (5 POINTS)
Mišánek has two cubic polynomials, F and G, which are monic[1]. He discovered that the following three equations have in total exactly 8 distinct solutions:

$$F(x) = 0,$$
$$G(x) = 0,$$
$$F(x) = G(x).$$

He wanted to see if the smallest and the largest of the numbers could both be a solution to the first equation. Prove that they can't.

PROBLEM 5. (5 POINTS)
For his birthday last year, Danil received a $(2n+1)$-tuple of non-zero integers $(k_0, \ldots, k_{2n})$ with non-zero sum. However, as everyone knows, Danil hates polynomials that have integer roots. So he rearranged the $(2n+1)$-tuple to create its permutation $(a_0, \ldots, a_{2n})$, such that the polynomial $a_{2n}x^{2n} + ... + a_0 = 0$ didn't have any integer roots. Prove that no matter what tuple Danil got, he was always able to find such a permutation.

PROBLEM 6. (5 POINTS)
Solve the following cyclic system of equations in variables x, y, z:

$$y = \frac{3x^3 + 4x}{x^2 + 12}, \qquad z = \frac{3y^3 + 4y}{y^2 + 12}, \qquad x = \frac{3z^3 + 4z}{z^2 + 12}.$$

[1] A polynomial $a_n x^n + \ldots a_1 x + a_0$ is called monic if $a_n = 1$.

Problem 7. (5 points)

Rado has four distinct non-zero numbers a, b, c, d. He found out that these numbers satisfy the following two equations:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4,$$
$$ac = bd.$$

Now he got curious what the largest value of the following expression could be:

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b}.$$

Help him find it.

Problem 8. (5 points)

Find all funtions $f : \mathbb{R} \to \mathbb{R}$ such that all $x, y \in \mathbb{R}$ satisfy the following equation:

$$(y + 1)f(x) + f(xf(y) + f(x + y)) = y.$$

Equations

4^{TH} AUTUMN SERIES MODEL SOLUTIONS

Problem 1.

Anička wrote $1 \square 2 \square 3 \square 4 \square 5 = 5$ on a blackboard. Then E.T. came along and wrote either a plus sign or a minus sign (of his choice) into one of the squares on the left side of the equation. Could E.T., being a pest, make it so that Anička cannot make the equation hold by filling the remaining three gaps with plus or minus signs?

(Anička Doležalová)

SOLUTION:
Since 1 is always positive (there is no box in front of it), we can subtract it from both sides. Suppose now that E.T. writes a minus sign in front of the number 4. Then we are left with $\square 2 \square 3 \square 5 = 8$. If we try to fill all the remaining boxes with pluses, we get 10, which is too much. However, if we put a minus in front of even the smallest of these numbers (which is 2), we can only get up to 6, which is too low. Therefore, if E.T. writes a minus sign in front of 4, Anička is at loss and cannot correct the equation.

POZNÁMKY:
Úloha se dala řešit mnoha způsoby, nejjednodušším z nich bylo vypsat všech šestnáct možností (resp. osm při správném tipu, že před čtyřkou musí být mínus). Složitější způsoby pak obsahovaly modulení. Ráda bych připomněla, že čím lehčí úloha, tím víc je potřeba vysvětlit i ty jednoduché věci, které v řešení osmičky rozebírat nemusíte.

(Anička Doležalová)

Problem 2.

Solve for $x \in \mathbb{R}$:

$$|x| + |x+1| + \cdots + |x+2018| = x^2 + 2018x - 2019.$$

(Rado van Švarc)

SOLUTION:
The left hand side of the equation is a sum of absolute values, therefore it is positive for every x. The right hand side is equal to $(x-1)\cdot(x+2019)$, which is positive only for $x \in (-\infty, -2019) \cup (1, \infty)$.

If $x > 1$ then all the sums inside absolute values are positive, therefore we can simplify the equation to

$$x + (x+1) + \cdots + (x+2018) = x^2 + 2018x - 2019,$$
$$2019x + 1 + \cdots + 2018 = x^2 + 2018x - 2019.$$

Because $1, 2, \ldots, 2018$ is an aritmetic sequence, we can easily get the sum of this sequence which then leads to:

$$2019x + 1009 \cdot 2019 = x^2 + 2018x - 2019,$$
$$0 = x^2 - x - 2019 \cdot 1010,$$
$$x = \frac{1 \pm \sqrt{8156761}}{2}.$$

One root, $x = \frac{1-\sqrt{8156761}}{2}$, is less than 1 which contradicts our initial assumption $x > 1$, so we are left with only one solution

$$x = \frac{1+\sqrt{8156761}}{2}.$$

For $x = \frac{1+\sqrt{8156761}}{2}$ all the operations above are equivalent, therefore this x is a solution of the problem.

If $x < -2019$, then all the sums inside absolute values are negative, therefore we can simplify the equation to:

$$\begin{aligned} -x-(x+1)-\cdots-(x+2018) &= x^2+2018x-2019, \\ -2019x-1009\cdot 2019 &= x^2+2018x-2019, \\ 0 &= x^2+4037x+2019\cdot 1008, \\ x &= \frac{-4037\pm\sqrt{8156761}}{2}. \end{aligned}$$

But $\frac{-4037+\sqrt{8156761}}{2} > -2019$, hence for $x < -2019$ there is again only one solution

$$x = \frac{-4037-\sqrt{8156761}}{2}.$$

For $x = \frac{-4037-\sqrt{8156761}}{2}$ all the operations above are equivalent, therefore this x is a solution of the problem.

Poznámky:

Většina odevzdaných řešení byla správná. Nejčastější chybou byla snaha sečíst levou stranu pomocí vzorečku $\frac{|2x+2018|\cdot 2019}{2}$ nebo jiným způsobem sečíst řadu absolutních hodnot, a to nezávisle na velikosti x. Bohužel, takhle jednoduše to nelze. *(„madam Verča“ Hladíková)*

Problem 3.

Pavel and Hedvika went for a walk by the seashore and found five cute baby seals. Because Pavel is a proper mathematician, he assigned a positive real number to each seal in order to quantify their cuteness. Then Hedvika noticed, that the product of cutenesses of any pair of baby seals is equal to the sum of cutenesses of the other three seals. What are the possible values of cutenesses of the five baby seals?

(Rado van Švarc)

Solution:

Let $x_1, x_2, x_3, x_4, x_5 > 0$ be the cutenesses of the seals. Using this notation, Hedvika gave us the following system of equations:

$$x_i x_j = x_k + x_\ell + x_m,$$

where (i, j, k, l, m) ranges over all permutations of $(1, 2, 3, 4, 5)$. We will show that the only solution to this system with positive numbers is $x_1 = x_2 = x_3 = x_4 = x_5 = 3$. The following are three possible ways of proving that this is the only solution (in the end one must always check whether the obtained numbers really solve the original problem, i.e. in our case one must simply check that $3 \cdot 3 = 9 = 3 + 3 + 3$).

First solution (adding equations):
Take the following four of the equations:

$$x_1x_2 = x_3 + x_4 + x_5,$$
$$x_1x_3 = x_2 + x_4 + x_5,$$
$$x_1x_4 = x_2 + x_3 + x_5,$$
$$x_1x_5 = x_2 + x_3 + x_4.$$

Adding all of them together yields $x_1(x_2+x_3+x_4+x_5) = 3(x_2+x_3+x_4+x_5)$. Since $x_2+x_3+x_4+ +x_5 > 0$, we can divide by it and get $x_1 = 3$. Similarly, we can deduce $x_2 = x_3 = x_4 = x_5 = 3$.

Second solution (subtracting equations):
By subtracting the following two equations

$$x_1x_2 = x_3 + x_4 + x_5,$$
$$x_1x_3 = x_2 + x_4 + x_5,$$

we get $x_1(x_2 - x_3) = x_3 - x_2$, thus $(x_1 + 1)(x_2 - x_3) = 0$. Since we know $x_1 + 1 > 0$, necessarily $x_2 = x_3$. Similarly, we obtain $x_1 = x_2 = x_3 = x_4 = x_5$. Let us denote this common value x. Now the problem reduces to finding the roots of one equation $x^2 = 3x$, or $x(x - 3) = 0$ which has only one positive solution $x = 3$.

Third solution (ordering of the variables):
Assume WLOG[1] that $x_1 \le x_2 \le x_3 \le x_4 \le x_5$ (otherwise we can relabel the variables so that the inequality holds). Then we get

$$x_5x_4 = x_1 + x_2 + x_3 \le x_4 + x_4 + x_4 = 3x_4, \text{ so } x_5 \le 3 \text{ (thanks to } x_4 > 0).$$

Similarly,

$$x_1x_2 = x_3 + x_4 + x_5 \ge x_2 + x_2 + x_2 = 3x_2, \text{ so } x_1 \ge 3.$$

Hence $x_1 = x_2 = x_3 = x_4 = x_5 = 3$.

Poznámky:
Sešla se spousta správných řešení, víceméně všechna se držela jednoho ze tří zmíněných způsobů. Pár řešitelů si však zřejmě v zadání nevšimlo slovíčka „positive“ a řešilo tak soustavu rovnic v reálných číslech, kde existují ještě další řešení: $(0, 0, 0, 0, 0)$ a $(3, -1, -1, -1, -1)$ a jejich permutace. Jelikož se jedná o těžší úlohu (a její řešení obsahuje řešení úlohy původní), body jsem za toto opomenutí nestrhával.

(Tonda Češík)

Problem 4.

Mišánek has two cubic polynomials, F and G, which are monic[2]. He discovered that the following three equations have in total exactly 8 distinct solutions:

$$F(x) = 0,$$
$$G(x) = 0,$$
$$F(x) = G(x).$$

He wanted to see if the smallest and the largest of the numbers could both be a solution to the first equation. Prove that they can't.

(Rado van Švarc)

[1] Without loss of generality (české BÚNO).
[2] A polynomial $a_nx^n + \dots a_1x + a_0$ is called monic if $a_n = 1$.

Solution:
Define $H(x) = F(x) - G(x)$ so that the solutions of the third equation $F(x) = G(x)$ are exactly the roots of the polynomial $H(x)$. As both F and G are monic, the cubic term x^3 cancels out and H is at most quadratic, hence it has at most two roots. We know that the equations have eight solutions altogether, so F and G both have three distinct roots and H has two.

Now let s be the smallest and t the largest of these eight roots. We will prove the claim by contradiction, so assume $F(s) = F(t) = 0$.

A monic cubic polynomial $(x-a)(x-b)(x-c)$ with three distinct roots $a < b < c$ is negative for all $x < a$ because all three brackets are negative for such x. Similarly, it is positive for all $x > c$. Therefore $F(s) = 0 > G(s)$ and $F(t) = 0 < G(t)$ and as H does not have any roots outside $[s,t]$[3], $F(x) > G(x)$ for $x < s$ and $F(x) < G(x)$ for $x > t$.

A quadratic polynomial with two distinct roots $d < e$ is either positive or negative for all $x \notin [d,e]$. Roots of H are inside $[s,t]$ so we have either $F(x) - G(x) > 0$ for $x \notin [s,t]$, or $F(x) - G(x) < 0$ for $x \notin [s,t]$, both of which contradict our previous statement that $F(x) > G(s)$ for $x < s$ and $F(x) < G(x)$ for $x > t$. Therefore, at least one of s, t is not a root of F, as we wanted to prove.

Poznámky:
Správná byla většina řešení, zhruba polovina řešitelů úlohu velmi elegantně sepsala, za což bych je chtěla pochválit. Některým řešením naopak chyběla pořádná struktura důkazu. Možná to bylo jen kvůli angličtině, ale rozhodně pro příště všem doporučuji zkontrolovat, jestli u důkazu sporem na začátku jasně napsali, co předpokládají, a jestli je u jednotlivých kroků zřejmé, z čeho plynou.

(Bára Kociánová)

Problem 5.

For his birthday last year, Danil received a $(2n+1)$-tuple of non-zero integers $(k_0, \ldots, k_{2n})$ with non-zero sum. However, as everyone knows, Danil hates polynomials that have integer roots. So he rearranged the $(2n+1)$-tuple to create its permutation $(a_0, \ldots, a_{2n})$, such that the polynomial $a_{2n}x^{2n} + ... + a_0 = 0$ didn't have any integer roots. Prove that no matter what tuple Danil got, he was always able to find such a permutation.

(Rado van Švarc)

Solution:
Let $(k_0, \ldots, k_{2n})$ be the $(2n+1)$-tuple Danil got for his birthday.

First we can set $(a_0, \ldots, a_{2n})$ as a permutation of $(k_0, \ldots, k_{2n})$ such that $|a_{2n}| \geq |a_i|$ for all i with $0 \leq i \leq 2n$. Now we will show that if $|x| \geq 2$, x cannot be a root of the polynomial $a_{2n}x^{2n} + \cdots + a_0 = 0$. If it was, then the equality $a_{2n}x^{2n} = -(a_{2n-1}x^{2n-1} + \cdots + a_0)$ would be true. But since $|x| - 1 \geq 1$, we have (by the triangle inequality)

$$\begin{aligned}
\left|\sum_{i=0}^{2n-1} a_i x^i\right| &\leq \sum_{i=0}^{2n-1} |a_i x^i|, \\
&= \sum_{i=0}^{2n-1} |a_i||x|^i, \\
&\leq |a_{2n}| \sum_{i=0}^{2n-1} |x|^i, \\
&\leq |a_{2n}| \frac{|x^{2n}| - 1}{|x| - 1}, \\
&< |a_{2n}x^{2n}|,
\end{aligned}$$

[3]English literature writes a closed interval between a and b using square brackets $[a,b]$ instead of the angle brackets $\langle a,b\rangle$ you are used to.

which is a contradiction, hence $|x| \geq 2$ cannot be a root.

Therefore, if x is an integer root, it must equal -1, 0, or 1. Moreover, since each coefficient is non-zero, 0 is not a root, and since the sum of all a is non-zero, 1 is not a root either. Hence the only possible root is -1.

If -1 is not a root, we have found Danil his beloved polynomial without any integer roots and we are done. So now suppose -1 is a root. Then $a_0 + a_2 + \cdots + a_{2n} = a_1 + a_3 + \cdots + a_{2n-1}$. If the polynomial is constant (i.e. $n = 0$), or if $a_0 = \cdots = a_{2n-1}$, we get $a_{2n} = 0$ which is a contradiction with every coefficient being non-zero. Hence we can find i with $0 < i < 2n$ such that $a_i \neq a_{i-1}$. Then $a_0 + \cdots + a_i x^{i-1} + a_{i-1} x^i + \cdots + a_{2n} x^{2n}$ gives a polynomial such that -1 is not its root. And since we still have $|a_{2n}| \geq |a_i|$ for all i with $0 \leq i \leq 2n$, this polynomial does not have any integer roots.

Poznámky:

Většina došlých řešení byla správně, občas někdo zapomněl na pár detailů jako zkontrolovat, že polynom není konstantní (pak hapruje logika, že můžeme najít $a_i \neq a_{i-1}$), nebo že neměníme a_{2n}, takže první část řešení stále platí. Obecně jsem za ně body nestrhával, ale když jich bylo víc, tak jsem body ubral.

U opravování této úlohy jsem si taky dost všímal angličtiny – ne gramatiky jako takové, ale matematického vyjadřování se. Takže když jsem našel nějaký termín, který se používá jinak, nebo formulaci, která jde říct hezčejc, navrhl jsem vám je. Za angličtinu jsem ale nikde body nestrhával, to není principem našeho semináře.

(Jáchym Solecký)

Problem 6.

Solve the following cyclic system of equations in variables x, y, z:

$$y = \frac{3x^3 + 4x}{x^2 + 12}, \qquad z = \frac{3y^3 + 4y}{y^2 + 12}, \qquad x = \frac{3z^3 + 4z}{z^2 + 12}.$$

(Jakub Löwit)

Solution:

We can rewrite the system as follows:

$$y = x\frac{3x^2 + 4}{x^2 + 12}, \qquad z = y\frac{3y^2 + 4}{y^2 + 12}, \qquad x = z\frac{3z^2 + 4}{z^2 + 12}.$$

Notice that the function $f(u) = \frac{3u^2+4u}{u^2+12}$ is positive for all real u. Hence either $x = y = z = 0$ or $x, y, z > 0$ or $x, y, z < 0$.

Triplet x, y, z is a solution if and only if $-x, -y, -z$ is also a solution because the function $f(u) \cdot u$ is odd. Therefore we can assume x, y, z are all positive. Then

$$f(u) \begin{cases} > 1 & \text{for } u > 2, \\ = 1 & \text{for } u = 2, \\ < 1 & \text{for } 0 < u < 2. \end{cases}$$

Let us solve the three cases:

(1) Suppose $x > 2$. This implies $y = x \cdot f(x) > x \cdot 1 = x$, so $y > x$. In particular we have $y > 2$. Then also $z = y \cdot f(y) > y$, so in turn $x = z \cdot f(z) > z$. But this yields a contradiction $x > z > y > x$.

(2) Let $x < 2$. Then we get $y = x \cdot f(x) < x \cdot 1 = x$, so $y < x$. But then $y < 2$, which yields $z = y \cdot f(y) < y$, so $x = z \cdot f(z) < z$, giving a contradiction $x < z < y < x$.

(3) Finally, if $x = 2$, then $y = x \cdot f(x) = x \cdot 1 = x$ and $z = y \cdot f(y) = y$. Plainly $x = y = z = 2$ is indeed a solution.

As we have already shown, a triplet (x, y, z) is a solution if and only if $(-x, -y, -z)$ is also a solution, so $(-2, -2, -2)$ is the only negative solution to the system.

The system has 3 solutions in total: $(x, y, z) = (0, 0, 0), (2, 2, 2), (-2, -2, -2)$.

Poznámky:

Většina řešení byla povedená. Mnoho myšlenek bylo shodných, to jest nějakým způsobem ukázat, že $f(u) = \frac{3u^3+4u}{u^2+12}$ je rostoucí. Občas přes rozebírání intervalů, občas přes derivace. Podle toho je provedené i vzorové řešení.

(Filip Čermák)

Problem 7.

Rado has four distinct non-zero numbers a, b, c, d. He found out that these numbers satisfy the following two equations:

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} = 4,$$

$$ac = bd.$$

Now he got curious what the largest value of the following expression could be:

$$\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b}.$$

Help him find it.

(Rado van Švarc)

Solution:

First of all, since $ac = bd$, we have $\frac{a}{b} = \frac{d}{c}$ and $\frac{b}{c} = \frac{a}{d}$. This means that if $x = \frac{a}{b} = \frac{d}{c}$ and $y = \frac{b}{c} = \frac{a}{d}$, we can rewrite the given condition as $x + y + \frac{1}{x} + \frac{1}{y} = 4$.

If x and y were both positive, we would have

$$0 = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + \left(\sqrt{y} - \frac{1}{\sqrt{y}}\right)^2.$$

This would imply $\sqrt{x} = \frac{1}{\sqrt{x}}$ and $\sqrt{y} = \frac{1}{\sqrt{y}}$, hence $x = 1$ and $y = 1$. That means $a = b$ and $b = c$, from which we also get $a = d$, from $ac = bd$. But this is a contradiction, since the numbers are not distinct. Therefore, x and y cannot both be positive.

Since $x + y + \frac{1}{x} + \frac{1}{y} = 4 > 0$, they cannot be both negative either. This means that one of them is positive and the other is negative. WLOG let $x < 0 < y$.

Since x is negative, $\sqrt{-x}$ exists, and we have $-x - 2 - \frac{1}{x} = \left(\sqrt{-x} - \frac{1}{\sqrt{-x}}\right)^2 \geq 0$, or $x + \frac{1}{x} \leq -2$. This also implies $y + \frac{1}{y} = 4 - x - \frac{1}{x} \geq 6$.

By combining the above equations, we get:

$$\begin{aligned}\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} &= \frac{a}{b} \cdot \frac{b}{c} + \frac{b}{a} \cdot \frac{a}{d} + \frac{c}{d} \cdot \frac{d}{a} + \frac{d}{c} \cdot \frac{c}{b}, \\ &= xy + \frac{y}{x} + \frac{1}{xy} + \frac{x}{y}, \\ &= (x + \frac{1}{x})(y + \frac{1}{y}), \\ &\leq (-2) \cdot 6, \\ &= -12.\end{aligned}$$

That means, that our maximum can be at most -12. On the other hand, if $a = -3 - \sqrt{8}$, $b = 3 + \sqrt{8}$, $c = 1$, $d = -1$, then obviously $ac = bd$ and a, b, c and d are distinct non-zero numbers. Also,

$$\begin{aligned}\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} &= \frac{-3-\sqrt{8}}{3+\sqrt{8}} + \frac{3+\sqrt{8}}{1} + \frac{1}{-1} + \frac{-1}{-3-\sqrt{8}},\\ &= (-1) + (3+\sqrt{8}) + (-1) + (3-\sqrt{8}),\\ &= 4,\end{aligned}$$

and

$$\begin{aligned}\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b} &= \frac{-3-\sqrt{8}}{1} + \frac{3+\sqrt{8}}{-1} + \frac{1}{-3-\sqrt{8}} + \frac{-1}{3+\sqrt{8}},\\ &= (-3-\sqrt{8}) + (-3-\sqrt{8}) + (-3+\sqrt{8}) + (-3+\sqrt{8}),\\ &= -12.\end{aligned}$$

So the value of -12 can be attained and therefore it is the maximum we were looking for.

Poznámky:
Mnoho řešitelů si nevšimlo slovíčka „distinct“ v zadání, díky čemuž řešili velice odlišnou úlohu.

Mezi 23 výsledky, které nám dorazily, se sedmkrát objevil správný výsledek -12, jedenáctkrát hodnota 4, třikrát -4, jednou 5 a jednou nic. *(Rado van Švarc)*

Problem 8.

Find all funtions $f : \mathbb{R} \to \mathbb{R}$ such that all $x, y \in \mathbb{R}$ satisfy the following equation:

$$(y+1)f(x) + f(xf(y) + f(x+y)) = y.$$

(Pavel Hudec)

Solution:
First note that the constant function $f(x) \equiv 1$ is clearly not a solution. Therefore there exists x_0 such that $f(x_0) \neq 1$. The equation has to hold for all x including x_0. Plugging in x_0 for x and rearranging yields:

$$f(x_0 f(y) + f(x_0 + y)) = y(1 - f(x_0)) - f(x_0).$$

We can see that the right hand side is a linear function in y and its range is $\mathbb{R}$. Therefore, the range of the left hand side has to be $\mathbb{R}$ as well. So we can conclude that f is surjective[4], hence there exists $a \in \mathbb{R}$ such that $f(a) = 0$.

If we set $x = 0$ in the original equation, we get:

$$\begin{aligned}(y+1)f(0) + f(f(y)) &= y,\\ f(f(y)) &= y(1 - f(0)) - f(0).\end{aligned}$$

We can show that $f(0) \neq 1$. If it was so, then we would have $f(f(y)) = -1$. But since f is surjective, $f \circ f$ is surjective as well. This is a contradiction with the fact that $f(f(y)) = -1$ for all real y.

Now we will prove that f is injective. Consider u and v such that $f(u) = f(v)$. Then we have the following chain of equalities:

$$u(1 - f(0)) - f(0) = f(f(u)) = f(f(v)) = v(1 - f(0)) - f(0).$$

Because of the earlier claim that $f(0) \neq 1$, we can conclude that $u = v$.

[4] A function $g : A \to B$ is surjective if its range is equal to its codomain B.

Having established surjectivity and injectivity, we can plug in $x = a$ and $y = 0$, where $f(a) = 0$:

$$\begin{aligned}(0+1)f(a) + f(af(0) + f(a)) &= 0,\\ f(af(0)) &= 0 = f(a).\end{aligned}$$

Because of injectivity, we can compare the inside parts and get equality of the arguments: $af(0) = a$. Using $f(0) \neq 1$, this is equivalent to $a = 0$. Hence $f(0) = 0$.

We will finish the solution by substituting $y = 0$ into the original equation:

$$f(x) + f(f(x)) = 0.$$

Combining this with $f(f(y)) = y(1 - f(0)) - f(0) = y$, we get $f(x) = -x$. It is easy to check that the function $f(x) = -x$ satisfies the equation.

Poznámky:

Úloha potrápila svou záludností i několik zkušených řešitelů. Zejména u funkcionálních rovnic doporučuji pečlivě si rozmyslet, jestli všechny kroky, které děláte, opravdu můžete udělat. Speciálně velkou část této úlohy tvořil důkaz vlastností funkce f, ty však někteří řešitelé používali bez důkazu.

(Pavel Hudec)