Integers

4^{TH} AUTUMN SERIES — DATE DUE: 8^{TH} JANUARY 2018

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)

Consider a pair of integers with the following properties:

(i) Their decimal representations do not contain zeros.
(ii) We can remove ten digits from the decimal representation of each of them to get identical numbers (not necessarily the same ten digits for both numbers).

Show that we can insert ten digits to the decimal representations of the two original numbers to obtain identical numbers.

PROBLEM 2. (3 POINTS)

A knight encountered a hundred-headed dragon. A sign in front of the cave says that knight can cut only 3, 5 or 8 heads in one swing. If he cuts off 3 of its heads, 9 new will grow. If he cuts off 5 heads, 2 new will grow and if he cuts off 8 heads, 11 new heads will grow. However, if the dragon loses its last head, it dies and no heads will regrow. Prove that the knight cannot kill the dragon.

PROBLEM 3. (3 POINTS)

Sir Filip and Madam Verča each picked a positive integer and secretly told Štěpán what their choice was. Štěpán wrote the product of the two numbers on one piece of paper and their sum on another one. Then he picked one of the pieces and showed it to Filip and Verča. It read 462. Filip said that he cannot determine Verča's number. After considering Filip's answer for a while Verča still couldn't find his number. What is Verča's number?

PROBLEM 4. (5 POINTS)

Let $s(x)$ denote the sum of the digits in the decimal expansion of x. Find all positive integers n such that[1] $s(n!) = 9$.

PROBLEM 5. (5 POINTS)

Anička and Bára have two sacks, one with m balls and the other with n balls. They decided to play a game with the following rules: They will take turns with Anička starting. In each turn, the player has to either remove a ball from one or both of the sacks, or move a ball from one sack to the other. If a ball is moved from one sack to another, it cannot be moved back in the very next move by the other player. Whoever removes the last ball, wins. With respect to m and n, who has the winning strategy[2]?

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[1] For a positive integer n we define $n! = 1 \cdot 2 \cdot \ldots \cdot n$ and we call this number *factorial of n*.

[2] A player has a winning strategy if he can achieve a win regardless of the moves of the other player.

PROBLEM 6. (5 POINTS)

Let n be a positive integer. Suppose that we have a partition of all positive integers into n sets such that if two distinct numbers belong to the same set, so does their sum. What is the highest number which can be an element of a singleton[3]?

PROBLEM 7. (5 POINTS)

Find all pairs of positive integers (n, k) satisfying the equation

$$n^k = (n-1)! + 1.$$

PROBLEM 8. (5 POINTS)

Consider a 2018×2018 checkerboard covered with 2×1 rectangular tiles. Prove that it is possible to fill in all 1×1 squares with positive integers in such a way that:

(i) The sum of the two numbers on every tile is always the same.
(ii) Two neighbouring numbers, whose corresponding squares share a side, are coprime if and only if they belong to the same tile.

[3] *Singleton* is a set which contains exactly one element.