Functions

4^{TH} AUTUMN SERIES — DATE DUE: 3^{RD} JANUARY 2017

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
David found the quadratic function $f : \mathbb{R} \to \langle 0, \infty)$, $f(x) = x^2$ and a function $g : \langle 0, \infty) \to \mathbb{R}$. For each of the compositions $f \circ g$ or $g \circ f$ decide whether it may be injective.

PROBLEM 2. (3 POINTS)
Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) < f(n)$ for all positive integers n?

PROBLEM 3. (3 POINTS)
Find a function $f : \mathbb{Z} \to \mathbb{Z}$ such that $f^{(n)}$ has exactly n roots for all positive integers n.

PROBLEM 4. (5 POINTS)
Find all functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying $f(x) + 2f\left(\frac{1}{x}\right) = x$ for all $x \in \mathbb{R} \setminus \{0\}$.

PROBLEM 5. (5 POINTS)
Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a function that satisfies $f(x)f(yf(x)) = f(x+y)$ for all $x, y \in \mathbb{R}^+$. Show that f is nonincreasing.

PROBLEM 6. (5 POINTS)
Lucien had a dream about a nonzero polynomial P with nonnegative integer coefficients. If Áďa says an integer z, Lucien tells her the value $P(z)$. What is the lowest number of questions Áďa has to ask to be able to figure out what Lucien's polynomial is?

PROBLEM 7. (5 POINTS)
A function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is said to be *cruelstrict* if $f^{(f(k))}(k) = k$ for all positive integers k. Prove that any cruelstrict function has at least $P(n) + 1$ fixed points, where $P(n)$ is the number of primes in $(\sqrt{n}, n\rangle$.

PROBLEM 8. (5 POINTS)
Let $f : \mathbb{N} \setminus \{1\} \to \mathbb{R}$ be a function given by $f(n) = \sqrt{2\sqrt{3\sqrt{\ldots(n-1)\sqrt{n}}}}$ for all positive integers $n > 1$. Prove that f is bounded by 3.