Introduction to Functions

The topic of the 4^{th} autumn series is functions. In this text, we try to get you acquainted with them.

Functions

It is actually rather difficult to define a function formally. For us, it will be an object – a blackbox – that takes certain inputs (called *arguments*). A function always returns one output for each input. Functions are usually denoted by letters f, g, h, or F, G, H.

Let us assume we have a function f. The set of all its inputs is called the *domain* of f and it is usually denoted by $D(f)$. A *range* (or *image*) $\operatorname{Rng}(f)$ of f is the set of all possible outputs of f.

When we write $f\colon X \to Y$, we mean that f is a function with X being its domain and Y its *codomain* – a set that has $\operatorname{Rng}(f)$ as a subset. The codomain, however, does not have to be equal to the image. For instance, if we define a function $f\colon \mathbb{R} \to \mathbb{R}$ by the formula $f(x) = x^2$ then its codomain is all real numbers while its image is only non-negative real numbers. In this text (as well as in the problems), we will deal with real functions of a real variable, which means all domains and codomains of all functions will be given subsets of real numbers. By the equality of functions $f = g$ we mean that $D(f) = D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$. A number $x \in D(f)$ is said to be a *root* of f if $f(x) = 0$.

A crucial fact about functions is that they are not the formulae or expressions that define them. If we, for example, define $f, g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = (x-1)^2 + 2x - 1$ and $f(x) = x^2$, then $f = g$ even though the expressions defining them are different. Furthermore, a function does not have to be defined by a formula. Such function may be $h\colon \mathbb{R} \to \mathbb{Z}$ that assigns to each real number the number of sevens in its decimal representation if it is finite and -1 if it is infinite.

Basic Properties

Functions may have many interesting properties, some of which are so common or so interesting that they have their own name. In this text, we shall mention only those that are needed to understand the problems given in the 4^{th} autumn series. A function $f\colon X \to Y$ is called

(1) *injective* if for each $x, y \in X$ the following holds: if $x \neq y$ then $f(x) \neq f(y)$,

(2) *bounded* if there is a constant $C > 0$ such that $|f(x)| \leq C$ for all $x \in X$,
(3) *nonincreasing* if $f(x_1) \geq f(x_2)$ for every x_1, $x_2 \in X$, $x_1 \leq x_2$.

Function Composition

Considering two functions $f\colon X \to Y$ and $g\colon Y \to Z$, one might wonder, what happens when we apply function g to an output of f. This *composition* of functions creates a function $h\colon X \to Z$ defined for any $x \in X$ by $h(x) = g(f(x))$. Such composed function is often denoted by $g \circ f$. If $f\colon X \to X$, we may compose f with itself. If we compose it multiple times, we usually use the following notation: $f^{(n)} = \underbrace{f \circ f \circ \dots \circ f \circ f}_{n}$.

Functional Equations

A functional equation is an equation where the unknowns are functions. In other words, we are searching for a function with a given domain and codomain for which the given equality holds. There is no universal way to solve a functional equation. However, we usually assume that we have a solution F of the equation and by clever manipulation with the given equation, we find what properties F must have. When we finally discover what relation describes F, it is necessary to verify if it satisfies the equation. To show the proccess let us solve the following problem.

Problem. Find all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that for all reals x, y it holds that

$$f(xy) = yf(x). \tag{1}$$

Solution. Let us assume that a function $F\colon \mathbb{R} \to \mathbb{R}$ is a solution of the given equation. Then (1) has to hold for $f = F$ and $x = 1$, which means

$$F(y) = yF(1),$$

where $F(1)$ is a constant. Thus we get that F is a linear function, i.e. $F(y) = cy$. Now we must find out for which constants c – if for any – such F is a solution of the given equation. The only thing we need to verify is $F(xy) = yF(x)$. For a linear function F, we get $F(xy) = cxy = ycx = yF(x)$, so $F(y) = cy$ is the solution for all $c \in \mathbb{R}$.

As you can see in the problem, functional equation may have more than one solution (or none at all).

Functional equations and functions in general are a wide topic that cannot be summarized in such a short text. If you want to learn more about it you may search through our library[1] or archive[2]. Especially recommended is a thorough text devoted to functional equations[3] written in Czech by Vejtek Musil. You may also focus on a similar text[4] which was used as a model for this introduction.

[1] https://mks.mff.cuni.cz/library/library.php
[2] https://mks.mff.cuni.cz/archive/archive.php
[3] http://mks.mff.cuni.cz/library/FunkcionalniRovniceVM/FunkcionalniRovniceVM.pdf
[4] http://mks.mff.cuni.cz/archive/33/uvod3p.pdf