Areas and perimeters

4$^{\text{TH}}$ AUTUMN SERIES — DATE DUE: 4$^{\text{TH}}$ JANUARY 2016

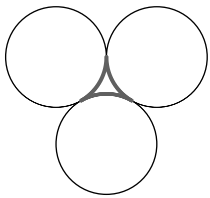
Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)

There are two unit squares. In the first one a circle is inscribed. The second one is divided into 49 congruent squares and in each of them a circle is inscribed. Decide what is bigger: the area of the circle in the first square, or the sum of the areas of all circles in the second one?

PROBLEM 2. (3 POINTS)

Three circles with perimeter 36 are given. Each two of them touch as shown in the picture. What is the perimeter of the area between the circles (grey in the picture)?



PROBLEM 3. (3 POINTS)

Divide a unit square into 2015 (not necessarily congruent) rectangles with perimeter 2.

PROBLEM 4. (5 POINTS)

The exterior angle at the vertex A of a triangle ABC equals 3φ. Let D, E be points on its sides AB and CA respectively, such that $|\sphericalangle ADC| = 2|\sphericalangle ABE| = 2\varphi$. Prove that the ratio of perimeters of triangles ADE and ABC is equal to $\frac{|AE|}{|AB|}$.

PROBLEM 5. (5 POINTS)

A convex quadrilateral $ABCD$ is given. Let M and N denote the midpoints of AB and CD respectively. Furthermore, let the intersection of AN and DM be X and the intersection of BN and CM be Y. Prove that the sum of the areas of triangles ADX and BCY is equal to the area of quadrilateral $MXNY$.

PROBLEM 6. (5 POINTS)

A triangle with heights h_1, h_2, h_3 and perimeter p is given. Prove that the inradius of the triangle with sides $1/h_1$, $1/h_2$, $1/h_3$ is equal to $1/p$.

Problem 7. (5 points)

Matěj drew a convex 2016-gon. Rado is curious about its area. They agreed Rado can choose two points on its perimeter and draw a line passing through them which splits the 2016-gon in two parts. Then Matěj tells Rado the smaller one of the areas of these parts. Is it enough for Rado to repeat this 2013 times to determine the area of the 2016-gon? To choose a point X on the perimeter means to choose two adjacent vertices A and B of the 2016−gon and the ratio $r \in \langle 0, 1\rangle$ such that $r = |AX|/|AB|$.

Problem 8. (5 points)

A unit square is cut into rectangles. Each of them is coloured by either yellow or blue and inside it a number is written. If the color of the rectangle is blue then its number is equal to rectangle's width divided by its height. If the color is yellow, the number is rectangle's height divided by its width. Let x be the sum of the numbers in all rectangles. Assuming the blue area is equal to the yellow one, what is the smallest possible x?