Trigonometric functions

4^{TH} AUTUMN SERIES — DATE DUE: 6^{TH} JANUARY 2015

Pozor, u této série přijímáme pouze řešení napsaná anglicky!

PROBLEM 1. (3 POINTS)
How many solutions of the equation $\sin(2014x) = 0$ are there for $x \in [0, \pi]$?

PROBLEM 2. (3 POINTS)
Show that every solution of the equation[1]

$$\tan\left(\sin^{2014} x\right) = \tan\left(\cos x^{2015}\right)$$

is also a solution of

$$\sin^{2014} x = \cos x^{2015}.$$

PROBLEM 3. (3 POINTS)
Find all solutions of $\sin x - \sqrt{3}\cos x = 2$, where $x \in [0, 2\pi)$.

PROBLEM 4. (5 POINTS)
Is there a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$|f(x+y) + \sin x + \cos y| < 2\,?$$

PROBLEM 5. (5 POINTS)
The internal angles α, β, γ of triangle PIG satisfy

$$(\sin\alpha + \sin\beta) : (\sin\beta + \sin\gamma) : (\sin\alpha + \sin\gamma) = 7 : 8 : 9.$$

Find the value of $\cos\alpha$.

PROBLEM 6. (5 POINTS)
A gadget has two buttons S and C and a display. At first, the display shows 1. If S (or C) is pressed when a number x is shown on the display, this number is rewritten to $\sin x$ (or $\cos x$). What is the minimum and the maximum value that can be displayed after 2015 presses? (Note that we can change the buttons during the process, so one can start with pressing C three times, then press S eight times and so on. Also assume that the gadget works with radians.)

PROBLEM 7. (5 POINTS)
Let x, y and z be positive real numbers such that $x + y + z = \pi/2$. Prove that

$$\cos(x-y)\cos(y-z)\cos(z-x) \geq 8\sin x \sin y \sin z.$$

[1] The notation $\sin^{2014} x$ means the same as $(\sin x)^{2014}$.

Problem 8. (5 points)

Phil constructed a black box. Given any real number x as an input, this magical device was able to show $\sin x$, $\cos x$, $\tan x$, $\arcsin x$, $\arccos x$, $\arctan x$ on its display as an output, as required by its operator.[2] Phil then started to play with his product. First he chose 0 as the input. After that, when the box showed him the output, he used this output number as the next input. Show that by cleverly choosing the functions that the black box used in each step, Phil could generate any non-negative rational number in a finite sequence of steps. Note that the black box works in radians.

[2] If the required operation could have been done with the number.